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ON COMPACT COMPLEX MANIFOLDS.

BY

S. BOCHNER, *Princeton University.*

[Received 6 September, 1947.]

In the present paper we will generalize a classical theorem (Theorem 1) from one to several complex variables and we are justifying this undertaking on two counts. Firstly, the mere question of finding a suitable wording for the desired generalization will emphasize the discrepancy between one and several variables, and secondly our proof for the k-dimensional statements will be new even for the classical case $k = 1$.

A fundamental difference between one and several variables is as follows. In the classical case the whole geometric theory is based on the comprehensive theory of uniformization, but for several variables no such theory of uniformization exists or is possible. This deficiency is responsible for many difficulties, but it is also productive of new view points and situations which the classical theory cannot evolve.

In some contexts a method for overcoming this deficiency is to introduce a so-called Hermitian metric, and our context will offer an opportunity for summarizing and correlating some recent results of the author's bearing on this method.

1. A Theorem for Curves of Genus $p > 1$.

The classical theorem which will be the subject of this paper is as follows.

Theorem 1. *On a closed Riemann surface of genus $p > 1$ there are only a finite number of conformal one-to-one tran formations of the surface into itself.**

* A 'conformal one-to-one transformation' is the same as a (complex) analytic homeomorphism'. We will not adhere to any rigid terminology.

A complete geometrical proof of this theorem, based on an idea of H. Poincare, will be found in the last chapter of H. Weyl's book, *Die Idee der Riemannschen Flæche.* Our proof will be a variant on that proof, but with considerable alterations. The theorem was originally due to H. A. Schwarz and M. Noether, and further illuminating comment, mainly in an algebraic direction, is due to Weierstrass, A. Hurwitz and Fatou.*

In the statement of the theorem the restriction $p > 1$ is very essential. Indeed for $p = 0$ a closed Riemann surface is a Gaussian sphere, and this surface admits infinitely many complex homeomorphisms. Each projective transformation

$$Z' = \frac{aZ+b}{cZ+d}$$

is one, and their totality is a continuous group of 3 complex or 6 real parameters. For $p = 1$, the surface is a double-torus of the kind encountered in the theory of elliptic functions and every such torus admits a two-dimensional group of translations. But for $p > 1$, the situation is radically different. Not only does there not exist a family of homeomorphisms which depends on a continuous parameter, but their totality cannot be infinite in number. The second half of this statement is by no means a formal consequence of the first half, but compare the remark following Theorem 3.

The only geometric property common to all surfaces with $p > 1$ and not present for $p = 0$ and $p = 1$ is the fact that the universal covering space is conformally equivalent with the unit circle

$$|t| < 1. \qquad (1)$$

Every geometric proof of Theorem 1 must therefore make

* In addition to the quotations in H. Weyl, see Appell et Goursat, *Fonctions Algebriques*, (Fatou's Second Edition), Chapters XI and XVI.

use of this fact and must obviously be somehow concerned with the group of transformations

$$t' = e^{i\vartheta}\frac{t-a}{1-\bar{a}t}, \qquad |a| < 1, \qquad (2)$$

which carries (1) into itself. This cannot be avoided. However in Weyl's chain of arguments there are many details concerning the specific nature of such transformations, valid only in one complex variable, and such details must be avoided if there is to be a generalization. We will indeed avoid them by appealing to some general results of recent date, valid for any number of variables, and in order to expound these we will next discuss the concept of a complex space.

2. Complex Manifolds. A space in k complex variables is a Hausdorff space of $2k$ real variables having the following property. Each point is contained in a neighbourhood which can be topologically mapped into a neighbourhood of the Euclidean space of k complex variables, and if two such neighbourhoods intersect then the specified Euclidean images of their intersection are being mapped into each other by a system of functions

$$z'_j = f_j(z_1, \ldots, z_k), \; j = 1, \ldots, k, \qquad (3)$$

which are complex-analytic in the variables given. Any such space is orientable.

We call S a compact (complex) manifold, if it is connected and is coverable by a finite number of its neighbourhoods. We are defining a manifold in this and no other way, although there is a strong heuristic tendency towards unifying this concept of compact manifold with that of an algebraic variety. An algebraic variety is the locus of a system of homogeneous polynomials

$$F_\lambda(z_1, \ldots, z_n) = 0, \; \lambda = 1, \ldots, l, \qquad (4)$$

in complex projective space of the variables $z_1 : z_2 : \cdots : z_n$. It has dimension $2(n-r-1)$, where r is the rank of the system. It is locally a manifold, except for "singularities". If there are no singularities, that is, if they are all locally

"uniformized", the variety is obviously a compact manifold. There is a classical conjecture that every algebraic variety is algebraically equivalent to another one on which no singularities are present, and on the other hand there are large classes of compact manifolds which can be algebraically generated as a variety. Thus, in all likelihood, there is no substantial difference between a compact manifold and a variety. But the possibility of such an identification will play no role in our reasoning, and for our part, we will be dealing exclusively with a manifold as previously defined.

3. Lie Groups of Homeomorphisms. Let

$$Q = f(P) \tag{5}$$

denote a complex homeomorphism of S, and let Γ denote the family of such homeomorphisms. If we take all topological (non-analytic) homeomorphisms, then they obviously form a group, since the succession of two and the inverse of any one, again belong to the family. In the case of complex-analytic transformations the succession of any two is again obviously complex-analytic. But this is not at all obvious for the inverse of a transformation, and it is actually not true for *real* analytic transformations. Thus, for instance, for the analytic homeomorphism $y = x^3$ of the real line into itself the inverse $x = y^{\frac{1}{3}}$ is not again analytic. But in the complex case the peculiarity just exhibited cannot arise* and an inverse does exist. This is related to the fact, with which it is virtually equivalent, that in the case of a complex one-to-one transformation, the Jacobian must be everywhere $\neq 0$. Thus Γ is a group. Due to the compactness of S this group can be easily topologized. To that purpose we take any distance $\rho(P_1, P_2)$ on S. Some such (non-analytic) metric can

* This point will be discussed at great length in Chapter VIII of the forthcoming book: S. Bochner and W. T. Martin, *Several Complex Variables*, Princeton University Press.

always be introduced. If $Q=f(P)$ and $Q=g(P)$ are any two elements of Γ, then the distance

$$\delta(f, g) = \max_{P \in S} \rho(f(P), g(P)) \tag{6}$$

obviously defines a topology in Γ, and the topology is equivalent for all distances $\rho(P_1, P_2)$. Now the first general theorem to which we will refer is as follows.

THEOREM 2. *The group* Γ *is a Lie group.*

We will not repeat the proof by which it was obtained* but only note the steps by which it was obtained. It was first concluded that Γ is locally compact. This conclusion was predicated on the two assumptions that S is compact, and that it is complex. But once it is known that Γ is locally compact, both assumptions can be discarded and Theorem 2 follows from the following theorem. *If a locally compact group* Γ *acts on a differentiable (locally Euclidean) space, then the group* Γ *is a Lie group.*

In this connection the term 'Lie group' designates a group which is not necessarily connected and possibly is zero-dimensional. If zero-dimensional it is countable, finite or infinite, and "totally discrete", meaning that for all pairs of distinct elements the distance is uniformly bounded away from zero.

If the group is not zero-dimensional, it has group generators, and such a generator is nothing else but a *contravariant* complex-analytic vector-field

$$\zeta^{\alpha} = \zeta^{\alpha}(z_1, \ldots, z_k), \ \alpha = 1, \ldots, k, \tag{7}$$

which is defined and regular everywhere in S. We point out that for us a vector does not cease to be regular if its components vanish at a point. Being "contravariant" means that for a change of coordinates

$$z'_{\alpha} = z'_{\alpha}(z_1, \ldots, z_k), \tag{8}$$

* *See*, S. Bochner and D. Montgomery, Locally compact groups of differentiable transformations, *Annals of Mathematics*, 47 (1946), 653.

the new components are connected with the old ones by the relations

$$\overset{\frown}{\zeta}^{\alpha} = \zeta^{\beta} \frac{\partial z'_{\alpha}}{\partial z_{\beta}} \tag{9}$$

(summation over β). Thus altogether we can draw from Theorem 2 the following conclusion.

THEOREM 3. *If S is any compact complex manifold then (i) either there exists on S a contravariant complex analytic vectorfield ζ^{α} which is not identically* 0, *or (ii) the totality of all complex homeomorphisms is totally discrete.*

Before proceeding with the subject matter proper we will state a few

CONJECTURES. (α) We will prove under special assumptions that there exists no vector field ζ^{α} and then, again using assumptions, prove that Γ is finite. The second step is probably unnecessary. It is probably true, that whenever on a compact complex S the entire group Γ is discrete, it is automatically finite.

(β) It is possible to represent S as a complex analytic product $S_1 \times S_2$, where S_1 is a complex multi-torus (in several variables) and the group of complex homeomorphisms of S_2 is semi-simple.

(γ) If a separable complex space S (not necessarily compact) is fibred without any singularities by a compact complex manifold S_1, then S is the direct product of S_1 with the base space.

4. COVERING SPACES. Let S be our compact manifold. We introduce its universal covering space denoting it by $\overset{\frown}{S}$. The fundamental group will be denoted by Δ and its elements by

$$u^0, u^1, u^2, \ldots, \tag{10}$$

the sequence is finite or infinite, and each u^n is a complex homeomorphism of $\overset{\frown}{S}$ without a fixed point. With each point $\overset{\frown}{P}$ of $\overset{\frown}{S}$ we can associate the sequence of points

$\overset{\hookrightarrow}{P}^n = U^n(\overset{\hookrightarrow}{P})$, $n = 0, 1, 2, \ldots$ Any two points of the sequence will be called equivalent. The original manifold S can be identified with the space of all sequences

$$P = \{\overset{\hookrightarrow}{P}^0, \overset{\hookrightarrow}{P}^1, \overset{\hookrightarrow}{P}^2, \ldots\} \tag{11}$$

in a suitable manner. For any given sequence (11) we will say that each $\overset{\hookrightarrow}{P}^n$ "covers" P or "lies over" P, and conversely we will say that P is a "projection" of $\overset{\hookrightarrow}{P}^n$. It is possible to choose a subset S^0 of $\overset{\hookrightarrow}{S}$ having the following property. The point-sets

$$S^n \equiv U^n\ (S^0) \quad , n = 0, 1, 2, \ldots \tag{12}$$

are mutually disjoint, and

$$\overset{\hookrightarrow}{S} = S^0 + S^1 + S^2 + \ldots. \tag{13}$$

The set S^0 is then called a fundamental set. It can be chosen in many special ways. We will only require that its closure in $\overset{\hookrightarrow}{S}$ shall be compact in $\overset{\hookrightarrow}{S}$, and such a choice is certainly possible.

An analytic homeomorphism

$$\overset{\hookrightarrow}{Q} = \overset{\hookrightarrow}{f}\,(\overset{\hookrightarrow}{P}) \tag{14}$$

of $\overset{\hookrightarrow}{S}$ will be called *periodic* if it carries equivalent points $\overset{\hookrightarrow}{P}$ into equivalent points $\overset{\hookrightarrow}{Q}$. By projection on to S it gives rise to a transformation

$$Q = f(P) \tag{15}$$

and it is easy to see that the latter is a homeomorphism of S. If u^r and u^s are any two elements of (10) then

$$u^r \overset{\hookrightarrow}{f} u^s \tag{16}$$

has the same projection as (14) and it is not hard to see that homeomorphisms (16) are the only ones having this property. But the important statement is as follows.

LEMMA 1. *Conversely every homeomorphism* (15) *has a*

periodic extension (14), *and if we take any two points* $\overset{\hookleftarrow}{A}$, $\overset{\hookleftarrow}{B}$ *with projections* A, B *such that*

$$B = f\,(A) \tag{17}$$

then there exists a periodic extension which satisfies the initial condition

$$\overset{\hookleftarrow}{B} = \overset{\hookleftarrow}{f}\,(\overset{\hookleftarrow}{A}). \tag{18}$$

PROOF. Introduce again the set (11), and another set

$$Q = (\overset{\hookleftarrow}{Q}{}^0,\ \overset{\hookleftarrow}{Q}{}^1,\ \overset{\hookleftarrow}{Q}{}^2, \ldots). \tag{19}$$

If (14) is to cover (15), then for every point $\overset{\hookleftarrow}{P}{}^m$ with projection P, the image point $\overset{\hookleftarrow}{Q} = \overset{\hookleftarrow}{f}\,(\overset{\hookleftarrow}{P}{}^m)$ must be suitably chosen among the points (19) in such a manner that for near-by points $\overset{\hookleftarrow}{P}$ the images $\overset{\hookleftarrow}{Q}$ shall also be near-by. From this it follows that if the desired transformation (14) has already been established at a point $\overset{\hookleftarrow}{P}$, it is constructible and uniquely determined in a neighbourhood of $\overset{\hookleftarrow}{P}$. Thus starting with the initial data $\overset{\hookleftarrow}{A}$, $\overset{\hookleftarrow}{B}$ we can construct (14) in some neighbourhood $\overset{\hookleftarrow}{N}\,(\overset{\hookleftarrow}{A})$ of $\overset{\hookleftarrow}{A}$. Also, if another covering of (15) with the same initial condition (18) has been constructed on a larger neighbourhood $\overset{\hookleftarrow}{N}_1 \supset \overset{\hookleftarrow}{N}$, then for the smaller neighbourhood $\overset{\hookleftarrow}{N}\,(\overset{\hookleftarrow}{A})$ the two must be identical. Therefore starting with the initial neighbourhood $\overset{\hookleftarrow}{N}\,(\overset{\hookleftarrow}{A})$ we can construct (14) in a maximal connected open subset $\overset{\hookleftarrow}{N}_1$ of $\overset{\hookleftarrow}{S}$. Since $\overset{\hookleftarrow}{S}$ is connected, if $\overset{\hookleftarrow}{N}_1$ were not all of $\overset{\hookleftarrow}{S}$, it would have a boundary point $\overset{\hookleftarrow}{P}$ in $\overset{\hookleftarrow}{S}$. However by continuity it would then be possible to continue (14) uniquely into a neighbourhood of $\overset{\hookleftarrow}{P}$ thus contradicting the assumption that $\overset{\hookleftarrow}{N}_1$ was maximal.

Thus (14) exists everywhere. Similarly, the inverse $P = f^{-1}(Q)$ with the inverse initial condition $\overset{\hookleftarrow}{A} = \overset{\hookleftarrow}{f}^{-1}(\overset{\hookleftarrow}{B})$ has a periodic extension

$$\overset{\hookleftarrow}{P} = \overset{\hookleftarrow}{f}^{-1}(\overset{\hookleftarrow}{Q}) \tag{20}$$

and the succession of (14) and (20) projects on the identity. From this it follows that (14) is a homeomorphism of $\overset{\hookleftarrow}{S}$.

Also all extensions (14) for all initial conditions form again a group.

5. Proof of Theorem 1.

We will base it on Theorem 3, first ruling out possibility (i) for $p > 1$. The non-existence of vector-fields was recently proved from curvature assumptions* and the argument will be restated afterwards, but for the present, we will give a much simpler reasoning.

Any scalar function $z = \phi(P)$ on S gives rise to a periodic function on $\overset{\hookleftarrow}{S}$ by putting

$$\phi(\overset{\hookleftarrow}{P^n}) = \phi(P),\ n = 0, 1, 2, \ldots \tag{21}$$

In a similar fashion, all vector fields, tensor fields etc., on S can be periodically extended onto $\overset{\hookleftarrow}{S}$. Now consider a periodic analytic contravariant vectorfield on $\overset{\hookleftarrow}{S}$. On the one hand, we have at present $k = 1$ and thus only one component of the vector exists, and on the other hand, we can cover $\overset{\hookleftarrow}{S}$ by the one coordinate system (1). Thus our vectorfield will be described by *one* complex analytic function in (1). We denote it by $\zeta(t)$. If the element u^n of Δ has the form

$$u^n t = e^{i\vartheta_n} \frac{t - a_n}{1 - \bar{a}_n t},\ |a_n| < 1, \tag{22}$$

then for a *scalar* $\phi(t)$, periodicity would mean

$$\phi(u^n t) = \phi(t).$$

* S. Bochner, Vector Fields and Ricci Curvature, *Bull. Amer. Math. Soc.*, 52 (1946), 776-97.

However, for our vector component $\zeta(t)$ a judicious application of relation (9) leads to the functional equation

$$\zeta(u^n t) = \zeta(t) \frac{d(u^n t)}{dt} \tag{23}$$

$$= \frac{\zeta(t) \, . \, e^{i\vartheta_n} \, (1 - a_n \bar{a}_n)}{(1 - \bar{a}_n t)^2}$$

Since for the origin $t = 0$, the images $u^n 0 = -e^{i\vartheta_n} a_n$ disperse towards the boundary $|t| = 1$, as $n \to \infty$, we have

$$|a_n| \to 1. \tag{24}$$

Since the closure of S^0 is compact in (1), the value of the factor

$$\frac{e^{i\vartheta_n} \, (1 - \bar{a}_n a_n)}{(1 - \bar{a}_n t)^2} \tag{25}$$

tends to 0 as $n \to \infty$, uniformly for t in S^0. From this it is easy to conclude that $\zeta(t)$ is bounded in (1) with boundary values 0. It must therefore be identically 0.

Thus, by Theorem 3, the group Γ on S is countable. Suppose it were infinite,

$$\Gamma = (T^0, T^1, T^2, \ldots). \tag{26}$$

By Lemma 1, each T^n is the projection of a homeomorphism $\overleftarrow{T}^n$ of (1). Denote the latter by

$$t' = \phi_n(t). \tag{27}$$

Since the closure of S^0 is contained in a circle $|t| \leqslant c < 1$ we may assume by Lemma 1 that

$$|\phi_n(0)| \leqslant c < 1.$$

A subsequence $\{ \phi_{n_p}(t) \}$ would then converge in such a manner that the projections T^{n_p} would converge uniformly in S. But then the group Γ would contain arbitrary small elements, contrary to the assumption of discreteness. Thus, Γ must be finite, as asserted in Theorem 1.

6. A Generalization to Several Variables.

Theorem 4. Assumptions. I. *Let* $\overset{\hookleftarrow}{S}$ *be a bounded domain in the complex variables* $t_1, \ldots, t_k$, *and let* (10) *be a group of its homeomorphisms which generates a compact manifold* S.

II. *If we denote the homeomorphism* u^n, $n = 0, 1, 2 \ldots$ *by*

$$t_\alpha' = \psi_{n\alpha}\,(t_1, \ldots, t_k),\ \alpha = 1, \ldots, k \tag{28}$$

then uniformly for $(t_1, \ldots, t_k)$ *in any closed subset* S' *of* $\overset{\hookleftarrow}{S}$ *the partial derivatives*

$$\frac{dt'_\alpha}{dt_\beta} = \frac{d\,\psi_{n\alpha}(t)}{dt_\beta},\ \alpha,\ \beta = 1, \ldots, k, \tag{29}$$

tend to 0, *uniformly as the image point* $(t_1', \ldots, t'_k)$ *approaches the boundary of* $\overset{\hookleftarrow}{S}$.

Conclusion. *The group of complex homeomorphisms of* S *is finite.*

Proof. A contravariant vector on $\overset{\hookleftarrow}{S}$ is described by k analytic functions in $\overset{\hookleftarrow}{S}$ with the periodicity properties

$$\zeta^\alpha\,(u^n t) = \zeta^\beta(t)\,\frac{\partial\psi_{n\alpha}(t)}{\partial\psi_\beta(t)},\ \alpha = 1, \ldots, k. \tag{30}$$

By Assumption II, each $\zeta^\alpha(t)$ is bounded with boundary values 0 in $\overset{\hookleftarrow}{S}$. A function in several complex variables assumes the maximum of its absolute values on the boundary, and thus $\zeta^\alpha(t) \equiv 0$.

Thus by Theorem 3, Γ is discrete. Each element T^n of Γ is the projection of a homeomorphism

$$t_\alpha' = \phi_{n\alpha}\,(t_1, \ldots, t_k)\cdot \alpha = 1, \ldots, k \tag{31}$$

of S and we may assume that for some point of $\overset{\hookleftarrow}{S}$ the image points $\{t'_\alpha\}$ lie in a compact subset of $\overset{\hookleftarrow}{S}$. If Γ were infinite, a subsequence of (31) would converge uniformly in the neighbourhood of every point of $\bar{S}_1$ and Γ would have arbitrarily small elements. Thus Γ is finite.

The most immediate special case is that of the sphere

$$\overset{\frown}{S} : |t_1|^2 + \ldots + |t_k|^2 < 1. \tag{32}$$

Its homeomorphisms can be written in the form

$$t'_\alpha = \frac{a_{\alpha 0} + \Sigma_\beta\, a_{\alpha\beta}\, z_\beta}{1 + \Sigma_\beta\, a_\beta\, z_\beta},\ \alpha = 1, \ldots, k, \tag{33}$$

where

$$\Sigma_\alpha |a_\alpha|^2 < 1 \tag{34}$$

$$\Sigma_\alpha |a_{\alpha 0}|^2 < 1 \tag{35}$$

$$\Sigma_\alpha\, a_{\alpha\beta}\, \bar{a}_{\alpha 0} = a_\beta,\ \beta = 1, \ldots, k \tag{36}$$

$$\Sigma_\alpha\, a_{\alpha\beta}\, \bar{a}_{\alpha\gamma} = a_\beta\, \bar{a}_\gamma,\ \beta \neq \gamma \tag{37}$$

$$\Sigma_\alpha |a_{\alpha\beta}|^2 - |a_\beta|^2 = 1 - \Sigma_\alpha |a_{\alpha 0}|^2,\ \beta = 1, \ldots, k. \tag{38}$$

If we introduce the $k+1$ vectors

$$A_{\alpha\rho} = (a_{1\rho}, a_{2\rho}, \ldots, a_{k\rho}),\ \rho = 0, 1, \ldots, k$$

and the inner product (A, B) of two vectors, then (35)-(38) can also be written as

$$(A_0, A_0) < 1 \tag{39}$$

$$(A_\beta, A_0) = a_\beta.\ \beta = 1, \ldots, k \tag{40}$$

$$(A_\beta, A_\gamma) = a_\beta\, \bar{a}_\gamma,\ \beta \neq \gamma \tag{41}$$

$$(A_\beta, A_\beta) - |a_\beta|^2 = 1 - (A_0, A_0). \tag{42}$$

It is easily seen that every such fractional transformation is a homeomorphism of (32), that they form a group, and that they are transitive on (32). Furthermore the classical Schwarz's lemma can then be suitably extended to show that there are no other homeomorphisms of (32) except these.

Since (33) transforms the point $(0, \ldots, 0)$ into the point A_0 whenever a variable transformation (33) moves the origin towards the boundary of (32), we have

$$(A_0, A_0) \to 1. \tag{43}$$

Now, at the origin we have

$$\frac{\partial t'_\alpha}{\partial t_\beta} = a_{\alpha\beta} - a_{\alpha 0}\, a_\beta,$$

and hence

$$\sum_{\alpha}\left|\frac{\partial t'_\alpha}{\partial t_\beta}\right|^2 = (A_\beta, A_\beta) - a_\beta (A_0, A_\beta) - \bar{a}_\beta (A_\beta, 0) + |a_\beta|^2 (A_\beta, A_0).$$

By (40), this is

$$\begin{aligned}&(A_\beta, A_\beta) - 2\,|a_\beta|^2 + |a_\beta|^2\ (A_0, A_0)\\ &= (A_\beta, A_\beta) - |a_\beta|^2 - |a_\beta|^2\,[1-(A_0, A_0)]\end{aligned}$$

and (42) finally leads to

$$(1-|a_\beta|^2)\ [1-(A_0, A_0)].$$

Thus (43) implies

$$\sum_{\alpha}\left|\frac{\partial t'_\alpha}{\partial t_\beta}\right|^2 \to 0,\ \beta = 1,\dots,k,$$

and this verifies the validity of Assumption II of Theorem 4, at least for the point $t = 0$. Its validity for other points uniformly in a compact set can be obtained by a more complicated estimate; or by a change of coordinates, of the form (33), which moves a variable point into the origin.

7. On assumption II in Theorem 4.

We will somewhat relax this assumption to suit a certain family of domains. The most typical representative of the family is the multicircle

$$|t_1| < 1, \dots, |t_k| < 1. \tag{44}$$

We denote its entire boundary by B. A boundary point arises if at least one inequality $|t_\alpha| < 1$ is replaced by an equality $|t_\alpha| = 1$. In particular, B contains the points

$$|t_1| = 1, \dots, |t_k| = 1; \tag{45}$$

the latter point-set will be denoted by B_0. For $k > 1$ it is a proper subset of B, but it has the following important property. If $f(t_1,\dots,t_k)$ is analytic in (44) and has boundary values 0 at the points of B_0 with no assumptions made for the points of $B-B_0$, it is nevertheless identically 0. This follows immediately from the iterated Cauchy formula

$$f(t_1,\dots,t_k) = \frac{1}{(2\pi)^k}\int_{C_1}\cdots\int_{C_k}\frac{f(\zeta_1,\dots,\zeta_k)\,d\zeta_1\dots d\zeta_k}{(\zeta_1-t_1)\dots(\zeta_k-t_k)}. \tag{46}$$

Now the homeomorphisms* of (44) are

$$t'_\alpha = e^{i\vartheta_\alpha} \frac{t_{p_\alpha} - a_\alpha}{1 - \bar{a}_\alpha t_{p_\alpha}}, \ |a_\alpha| < 1, \tag{47}$$

where $\alpha = 1, \ldots, k$ and $(p_1, \ldots, p_k)$ is a permutation of $(1, \ldots, k)$. Now, if $(t_1, \ldots, t_k)$ lies in a closed subset of (44), then for variable transformations (47) the partial derivatives

$$\frac{dt'_\alpha}{dt_\beta} \tag{48}$$

do not necessarily converge to 0 as the point $(t_1', \ldots, t'_k)$ approaches the boundary B, but they do so converge if it approaches the subset B_0. However owing to the property of B_0 just stated, this suffices to establish the conclusion of Theorem 4.

In general, let B denote the boundary of any bounded domain $\overset{\frown}{S}$. A subset B_0 of B shall be called *significant* if any analytic function in $\overset{\frown}{S}$ with boundary values 0 on B_0 is identically 0. Then in Assumption II of Theorem 4 it suffices to assume that the derivatives (29) converge to 0 as the variable point t' approaches the significant part only.

The domain (44) is a product of circles and the significant subset (46) is the product of their boundaries, which is a much smaller set than the boundary of the products.

More generally, whenever a domain $\overset{\frown}{S}$ is a product of domains

$$S_1 \times S_2 \times \ldots \times S_l \tag{49}$$

in lower dimensional factors, then the formal product of their boundaries

$$B_1 \times B_2 \times \ldots \times B_l \tag{50}$$

* Caratheodory, Ueber das Schwarzsche Lemma bei analytischen funktionen von zwei komplexen Veraenderlichen, *Math. Annalen*, 97 (1927), 98.

is a significant subset of the boundary of (49). This follows from the fact that there exists a Cauchy formula* for any domain S_λ, and in the case of a product (49), a formula can hence be obtained by iteration, roughly analogous to (46), which determines the values in (49) from those on (50).

However it must not be assumed that the factorization (49) is the main source for significant sets. They may arise from algebraic sources on domains which on the face of it are very much "irreducible". The best illustration for this are the so-called matrix spaces, and we will briefly describe one case. Assume that the dimension number k is a square, $k = n^2$. The variables

$$Z_1, \ldots, Z_k \tag{51}$$

can then be arranged as a square matrix

$$Z = \{ Z_{pq} \}, p, q = 1, \ldots, n. \tag{52}$$

The symbol Z shall denote the entire matrix. If Z^* denotes the adjoint matrix then the matrix product ZZ^* is positive Hermitian. The domain S shall be the set of points in the space (50) for which all eigen-values of ZZ^* are < 1. The boundary B consists of those points for which all eigen-values are $\leqslant 1$ with at least one eigen value strictly $= 1$, and the subset B_0 consists of those points for which all eigen-values are $= 1$. A matrix (51) belongs to (51) if and only if it is unitary. There exists a Cauchy formula†

* Due to R. Fueter and Enzo Martinelli. *See*, Martinelli, Alcumi teoremi integrali per le funzioni analitiche di pui variabili complesse, *Reale Acca Reale* (*Mat.*), 1938 XVI, p. 272. A thorough discussion of the formula with many implications was given in S. Bochner, Analytic and meromorphic continuation by means of Green's formula, *Annals of Mathematics*, 44 (1943), 652-73. However this author's claim in *loc. cit.* footnote (1), that he might have been familiar with the general shape of the formula before Martinelli, was wholly unjustified and is hereby being retracted.

† S. Bochner, Group invariance of Cauchy's formula in several variables, *Annals of Mathematics*, 45 (1944), 699.

$$f(z) = \frac{1!2!\dots(n-1)!}{(2\pi i)^{\frac{n(n+1)}{2}}} \int_{B_0} \frac{f(\zeta)\, d\zeta_1 \dots d\zeta_k}{(\det|\zeta_{pq} - z_{pq}|)^n} \tag{53}$$

from which it is obvious that B_0 is indeed a significant set.

8. NEGATIVE CURVATURE*

If we start from a manifold S, then hypothesis I in Theorem 4 implies the hypothesis that the covering space $\tilde{S}$ shall be complex-analytically equivalent with a domain in ordinary space. For $k \geqslant 2$ this hypothesis is implacably severe and wholly unverifiable from the definition of a manifold. In this direction nothing is known, not even in the simplest case, whether real analytic or complex analytic. For instance, if $\tilde{S}$ is topologically a simplex, and if analytically it is a union of two analytic neighbourhoods, each topologically a simplex and if their intersection is a simplex, then it is not known whether $\tilde{S}$ can be analytically covered by one coordinate system, that is, whether it can be topologically-analytically mapped into Euclidean space. Such a mapping requires the existence of certain analytic functions on all of $\tilde{S}$. Now except in the classical case, there is no means known of constructing an analytic function in the large, out of nothing as it were. Some global information must be supplied to start with. The most natural information of this kind is an analytic distance, and this implies the existence† of an analytic positive definite differential form $ds^2 = g_{ij}\, dx_i\, dx_j$ on the entire space. In the real case, this author was the first to prove, as a supplement to

* In this section we will partly presuppose, and partly summarize and draw inferences from the main facts in the author's paper mentioned in sec. 5, page 9, and the further paper, S. Bochner, Curvature in Hermitian metric, *Bulletin of American Math. Soc.*, 53 (1947), 179-95.

† *See* S. Bochner, Differentiable and Riemann Metric, *Duke Journal*, 4 (1938), 51-4.

H. Whitney's work on differentiable manifolds, that if on a compact analytic manifold such an analytic metric exists, the manifold can be topologically analytically imbedded into Euclidean space*.

In the complex case a natural restriction on the metric is to assume that it has the Hermitian form

$$ds^2 = g_{\alpha\bar{\beta}}\, dz_\alpha d\bar{z}_\beta. \tag{53}$$

A further restriction, very far reaching but very fruitful, is the further assumption

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z_\gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial z_\alpha} \tag{54}$$

which automatically implies

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{z}_\gamma} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{z}_\beta} \tag{55}$$

This amounts to assuming that locally there exists a scalar function $\Omega\ (z, \bar{z})$ such that

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \Omega}{\partial z_\alpha \partial \bar{z}_\beta}. \tag{56}$$

In this case the Riemann-Christoffel tensor has components only of the form

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} \tag{57}$$

and for the contracted Ricci-tensor $R_{\alpha\bar{\beta}}$, we then have the beautiful relation

$$R_{\alpha\bar{\beta}} = \frac{\partial^2 G}{\partial z_\alpha \partial \bar{z}_\beta}, \tag{58}$$

where G is the determinant of the matrix.

Whenever on a Riemannian space, the two-dimensional sectional Gaussian curvature is positive for all sections through a point, then of necessity the Ricci curvature is positive at that point, meaning that†

$$-R_{\alpha\bar{\beta}}\xi^\alpha\bar{\xi}^\beta > 0 \tag{59}$$

for any vector ξ^α of non-vanishing length. But not conversely. Thus to draw a conclusion from the behaviour

* S. Bochner, Analytic mapping of compact Riemann spaces into Euclidean space, *Duke Journal*, 3 (1937), 339-54.

† The negative sign is due to a conventional normalization.

of Ricci curvature is better than to conclude it from the behaviour of the Riemann curvature. We now quote the following theorem.

THEOREM 5. *If on a compact complex manifold the Ricci curvature is positive or zero (but not identically zero), then there exists no covariant complex analytic tensor other than zero.*

If however the Ricci curvature is negative or zero (but not identically zero) then there exists no contravariant analytic tensor other than zero.

It will also be useful to introduce an *ad-hoc* definition. We say that a space $\overset{\frown}{S}$ has the *Montel structure* if every sequence of its complex homeomorphisms for which the images of one special point are contained in a compact subset of $\overset{\frown}{S}$ contains a subsequence which converges uniformly in the neighbourhood of every point of $\overset{\frown}{S}$.

By the use of Theorem 3 and the second half of the proof of Theorem 4, we now obtain on the basis of Theorem 5 the following conclusion.

THEOREM 6. *If a compact complex manifold S has negative Ricci curvature and if its universal covering space has the Montel structure, then there are only a finite number of complex homeomorphisms on S.*

9. COMPARISON BETWEEN THEOREMS 4 AND 6.

We are now going to demonstrate that in substance (although by no means in detail) the assumptions in these two theorems are very much alike.

Suppose first the assumptions of Theorem 4 are fulfilled in detail. Then $\overset{\frown}{S}$, being an ordinary bounded domain, has Montel structure, and we still have to argue that S is likely to possess a metric with negative curvature; or, what would be even better, that $\overset{\frown}{S}$ itself is likely to possess a hyperbolic geometry (metric with negative curvature) which is invariant with respect to all homeomor-

phisms of $\overset{\frown}{S}$ and not only those of any one fundamental group.

Now the theory of automorphic functions is full of indications that such is indeed the case. First of all, for $k=1$, we have the classical hyperbolic geometry with

$$ds^2 = \frac{2\,dz\,d\bar{z}}{(1-z\bar{z})^2}$$

in the unit circle $|z|<1$, the curvature being $K=-1$. The more general line element

$$ds^2 = \frac{2dz\,d\bar{z}}{(1+\tfrac{1}{2}b\,z\bar{z})^2} \tag{60}$$

has curvature $\frac{1}{2}b$ and for positive b it is an elliptic geometry on the one-dimensional complex projective space (Gaussian sphere). The generalization of (60) to k complex variables is

$$ds^2/2 = \frac{\Sigma_\alpha |dz_\alpha|^2 + b/2\,(\Sigma_\alpha |z_\alpha|^2\, \Sigma_\beta |dz_\beta|^2 - |\Sigma_\alpha \bar{z}_\alpha dz_\alpha|^2)}{(1+b/2\,\Sigma_\alpha |z_\alpha|^2)^2} \tag{61}$$

due largely to Fubini.

The sectional curvature is not constant on all sections, but only on holomorphic sections; but on all sections we have

$$\tfrac{1}{4} \leqslant k/b \leqslant 1, \tag{62}$$

and thus for negative b the sectional curvature is at any rate everywhere negative. The constancy on holomorphic sections expresses itself in the relation

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -b/2\,(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}). \tag{63}$$

In recent years, C. L. Siegel and L. K. Hua* have considered, for several variables, other 'natural' types of domains $\overset{\frown}{S}$ in many matrix spaces, and in all cases proved the existence of a hyperbolic geometry. Finally this author has given a general argument, partly heuristic, for the existence of a hyperbolic geometry in the general case. The argument runs as follows. Start with the original metric

* Quotations in the papers of footnote on page 16, Sec. 8.

$$ds^{(o)2} = \Sigma_\alpha dt_\alpha \bar{dt}_\alpha.$$

If we desire invariance only with respect to the elements u^n of a fundamental group Δ, then with each element u^n we associate the transformed line-element

$$ds^{(n)2} = \Sigma_\alpha dt_\alpha^{(n)} \overline{dt_\alpha^{(n)}} = \Sigma_{\alpha,\beta} g_{\alpha\bar{\beta}}^{(n)} dt_\alpha \bar{dt}_\beta$$

and we then form the sum

$$ds^2 = \Sigma_n ds^{(n)2} \tag{64}$$

provided it is finite. If then (64) exists, it is obviously periodic; but we were also able to show that its sectional curvature on holomorphic sections and its Ricci curvature must indeed be always negative (non-positive). This was obtained from a seemingly different theorem, namely from the fact that in the case of a complex imbedding of one space into a higher dimensional one, the so-called relative parts of these curvatures are negative. In particular, if the higher dimensional space is flat, it contributes no local curvature of its own, and the relative parts of these curvatures constitute their entire amounts for the imbedded space. Finally forming the line element (64) as a sum can be interpreted, in the limit, as if resulting from imbedding of $\tilde{S}$ into a higher dimensional space, and the negative character of the curvatures is the result.

Thus far we have discussed the relation of Theorems 4 and 6 in one direction. In the opposite direction, suppose that the assumptions of Theorem 6 are fulfilled in detail. Assume, more strongly, that the sectional curvatures themselves, are negative. Then by a known theorem on differentiable manifolds*, the covering space $\tilde{S}$ is *topologically* homeomorphic to a simplex. The vital question remaining is whether $\tilde{S}$ has the Montel structure. This seems to be a difficult question indeed, but perhaps within reach of the present day methods and we still

* E. Cartan, *Leçons sur la geometrie des spaces de Riemann*, Chapter VI.

venture to advance a tenuous argument in favour of an affirmative answer.

In the real case, all spaces of constant curvature (positive or negative) are, locally, isometrically equivalent with each other, by a classical theorem of F. Schur. From this it follows, by an argument using geodesics* that their universal covering spaces are the same. In the Hermitian case we have proved, in direct analogy to the theorem of F. Schur, that relation (63) implies the existence of an analytic coordinate system in which the line element has the form (61). Starting from this, it ought to be possible to imitate the argument for the real case with the view to obtaining the following theorem.

THEOREM 7. *If a complex manifold S has an Hermitian metric with property* (54) *and if* (63) *holds, and if the space is complete (meaning that its geodesics are continuable indefinitely) in particular if it is compact, then its universal covering space is analytically isometrically equivalent with the unit sphere* $|z_1|^2+\ldots+|z_k|^2 < 1$ *for* $b = -2$, *and with the complex projective space for* $b > 0$.

In particular, for $b = -2$, $\overset{\frown}{S}$ would have the Montel structure. Now, it is our further surmise that the same conclusion ought then to be obtainable for any compact complex manifold S of negative sectional curvature. If the negative value of the sectional curvature lies between k_0 and k_1, $k_0 < k_1$, then on the basis of (62) one ought to be able geometrically to establish the Montel structure of $\overset{\frown}{S}$ by comparing $\overset{\frown}{S}$ with two Fubini spaces, with the constants

$$|b_1| = k_0/4, \quad |b_2| = 4\,|k|,$$

which, as it were, envelop $\overset{\frown}{S}$ from above and below, and have Montel structure themselves.

* *See* Heinz Hopf, Zum Clifford-Kleinschen Raumproblem, *Math. Annalen*, 95 (1925), 317-8.

CESARO SUMMABILITY OF A CLASS OF FUNCTIONS.

BY

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This note suggests a further use for a difference-formula of L. S. Bosanquet(1) which has already proved to be of service in dealing with certain problems involving Cesàro and Riesz means(4, 5, 6). More particularly, it is the object of the note to establish, by means of Bosanquet's formula, the integral analogue of a theorem on series, due to M. S. Macphail (3), [Theorem 2], which contains as a special case a result of H. L. Garabedian(2). Macphail's theorem itself can be obtained, as shown in the last section of the note, by an obvious adaptation of the argument used to prove its integral analogue.

1. The following theorem is built around the same central idea as Macphail's.

THEOREM A. *If* $\phi(x) = \Sigma_{r=0}^{k-1} \phi_r(x)\, x^r$, $\phi_r(x)$ *being periodic functions with the same period* ω *and such that the mean value of each* $\phi_r(x)$ *over a period is* 0, *then* $\phi(x)$ *is summable by Cesàro's* k *th mean, but not by the* $(k-1)$ *th mean, to the value* 0.

NOTATION. In the proof of Theorem A, I shall employ the following notation.

Let $S(x)$ be defined in $(0, \infty)$ and of bounded variation in every finite interval. Then its rth Cesàro mean, $\sigma_r(x)$, and its rth Cesàro sum, $S_r(x)$, are given by

$$\sigma_r(x) = \frac{S_r(x)}{x^r} = \frac{r}{x^r}\int_0^x (x-t)^{r-1} S(t)\, dt,\ r > 0.$$

Given a function of x, $F(x)$, and any $\omega > 0$, let us write

$$\Delta_\omega^k F(x) = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} F(x+\overline{k-\nu}\,\omega),\ k=1,\ 2,\ 3,\ldots$$

and further agree to set $\Delta_1^k F(x) = \Delta^k F(x)$, $F(x) = \Delta^0 F(x)$.

Bosanquet's formula can then be expressed as below and proved by induction.

LEMMA 1

$$\Delta_\omega^k S_k(t_0) = \Gamma(k+1)\int_{t_0}^{t_0+\omega} dt_1 \int_{t_1}^{t_1+\omega} dt_2 \int_{t_2}^{t_2+\omega} \cdots \int_{t_{k-1}}^{t_{k-1}+\omega} S(t)\, dt,$$
$$k = 1, 2, 3, \ldots$$

PROOF OF THEOREM A. It suffices to prove that if

$$S(x) = \phi(x) = f(x)x^r,\ r = 0,\ 1,\ 2,\ldots,\ k-1, \tag{1}$$

then

$$f(x+\omega) = f(x),\ \frac{1}{\omega}\int_0^\omega f(x)\, dx = 0, \tag{2}$$

$$\sigma_k(x) = \frac{S_k(x)}{x^k} = o\ (1),\ x \to \infty. \tag{3}$$

We begin by defining functions $F_i(x)$ and constants c_i, $i = 1,\ 2,\ 3,\ldots$, as follows.

$$f_1(x) = \int_0^x f(t)\,dt,\ c_1 = \frac{1}{\omega}\int_0^\omega f_1(t)\,dt,\ F_1(x) = f_1(x) - c_1,$$

$$f_2(x) = \int_0^x F_1(t)\,dt,\ c_2 = \frac{1}{\omega}\int_0^\omega f_2(t)\,dt,\ F_2(x) = f_2(x) - c_2,$$
$$\text{etc.,} \tag{4}$$

so that $F_i(x)$ is obviously subject to the same conditions as $f(x)$ in (2).

(3) can be proved by showing that

$$\Delta_\omega^k S_k(t_0) = \Gamma(k+1)\int_{t_0}^{t_0+\omega} dt_1 \int_{t_1}^{t_1+\omega} \cdots \int_{t_{k-1}}^{t_{k-1}+\omega} f(t)\, t^r dt = 0,$$
$$0 \leqslant r \leqslant k-1. \tag{5}$$

For, this relation implies that $S_k(t_0)$ = a polynomial in t_0 of degree $(k-1)$ + a function of period ω, whence (3) follows at once. Now an integration by parts gives

$$\int_{t_{k-1}}^{t_{k-1}+\omega} f(t)t^r dt = f_1(t_{k-1})\Delta_\omega^1 t_{k-1}^r - \int_{t_{k-1}}^{t_{k-1}+\omega} f_1(t) rt^{r-1} dt$$

$$= [F_1(t_{k-1})+c_1]\Delta_\omega^1 t_{k-1}^r - \int_{t_{k-1}}^{t_{k-1}+\omega} [F_1(t)+c_1]\ rt^{r-1}dt$$

$$= F_1(t_{k-1})\Delta_\omega^1 t_{k-1}^r - \int_{t_{k-1}}^{t_{k-1}+\omega} F_1(t) rt^{r-1} dt$$

and therefore repeated integrations by parts lead to the identity

$$\int_{t_{k-1}}^{t_{k-1}+\omega} f(t)t^r dt = F_1(t_{k-1})\Delta_\omega^1 t_{k-1}^r - F_2(t_{k-1})\Delta_\omega^1 r t_{k-1}^{r-1} + \ldots$$
$$+ (-1)^{r-1} F_r(t_{k-1})\Delta_\omega^1 r!\ t_{k-1},$$

where the typical term, on the right side is $g(t_{k-1})t_{k-1}^s$ $(0 \leqslant s \leqslant k-2)$, $g(t_{k-1})$ satisfying the conditions imposed on $f(t)$ in (2).

We have thus reduced the proof of (5) to that of

$$\int_{t_0}^{t_0+\omega} dt_1 \int_{t_1}^{t_1+\omega} \ldots \int_{t_{k-2}}^{t_{k-2}+\omega} g(t_{k-1})t_{k-1}^s dt_{k-1} = 0,\ 0 \leqslant s \leqslant k-2. \tag{5.1}$$

Consequently, by repeated reductions of this kind, we can complete the proof of (5) and therefore that of (3) as well.

The negative conclusion of the theorem emerges at the same time as the positive from the foregoing argument.

The next theorem is a deduction from Theorem A.

THEOREM B. *Let $f(x)$ satisfy (2) and let $F_i(x)$, c_i be defined by (4). Then the integral $\int_0^\infty f(x)a(x)dx$ in which $a(x)$ is a polynomial of degree $(k-1)$, is summable (C, k), but not $(C, k-1)$, to the value*

$$c_1 a(0) - c_2 a'(0) \ldots + (-1)^{k-1} c_k\ a^{(k-1)}(0).$$

To prove this we require the following lemma [cf. 3. Lemma 1], which can be readily proved by repeated integrations by parts:

LEMMA 2. *Under the conditions of Theorem B,*

$$\int_0^x f(t)\,a(t)\;dt=\sum_{i=0}^{k-1}(-1)^i c_{i+1}a^{(i)}(0)+\sum_{i=0}^{k-1}(-1)^i F_{i+1}(x)\,a^{(i)}(x), \tag{6}$$

where $c_i, F_i(x)$ are defined by (4) *and $a^{(0)}(x)=a(x)$.*

PROOF OF THEOREM B. In the proof of Theorem A, we can set $\phi(x)=\int_0^x f(t)\,a(t)\;dt$, this being justified by (6). The desired conclusion then follows immediately since

$$\begin{aligned}\Delta_\omega^k S_k(t_0)&=\Gamma(k+1)\int_{t_0}^{t_0+\omega}dt_1\int_{t_1}^{t_1+\omega}\cdots\int_{t_{k-1}}^{t_{k-1}+\omega}\phi(t)\,dt\\&=\Gamma(k+1)\,\omega^k\sum_{i=0}^{k-1}(-1)^i c_{i+1}a^{(i)}(0).\end{aligned}$$

REMARK. If, in Lemma 2, $a(x)$ is not a polynomial, the right-hand member of (6) will have the extra term

$$(-1)^k\int_0^x F_k(t)\;a^{(k)}(t)\;dt.^*$$

Hence, if $a(x)$ in Theorem B is not a polynomial but satisfies the condition

$$\int_0^\infty F_k(x)\,a^{(k)}(x)\;dx \textit{ is convergent,} \tag{7}$$

then the integral $\int_0^\infty f(x)\;a(x)\;dx$ is summable (C, k) to the value

$$\sum_{i=0}^{k-1}(-1)^i c_{i+1}a^{(i)}(0)+(-1)^k\int_0^\infty F_k(x)\;a^{(k)}(x)\;dx.$$

* To secure (7) one may suppose for instance that either $a^{(k)}(x)\downarrow 0$ as $x\to\infty$ or $\int_0^\infty |\,a^{(k)}(x)\,|\;dx$ is convergent.

2. It is now easy to state results analogous to Theorems A, B and Lemmas 1, 2, involving sums in place of integrals and finite differences in place of derivatives. In the proofs of these results we have to employ the device of partial summation instead of that of integration by parts resorted to in the preceding discussion.

LEMMA 1′. *In Lemma* 1, *let* $S(t) = S_n$ *when* $n \leqslant t < n+1$ $(n = 0, 1, 2, \ldots)$ *and* $\omega = p$, *a positive integer. Then*

$$\Delta_p^k S_k(n_0) = \Gamma(k+1) \sum_{n_0}^{n_0+p-1} \sum_{n_1}^{n_1+p-1} \cdots \sum_{n_{k-1}}^{n_{k-1}+p-1} S_n.$$

THEOREM A′. *Let* $f(n)$ *have the (integral) period* p *and mean value* 0 *over a period. Then the sequence* $f(n)\,n^r$, $0 \leqslant r \leqslant k-1$, *is summable* (C, k), *but not* $(C, k-1)$, *to the value* 0.

THEOREM B′ (Macphail). *Let* $a(n)$ *be a polynomial of degree* $(k-1)$ *in* n. *Let* $f(n)$ *be as in Theorem A′ and further let constants* c_i $(i = 1, 2, 3, \ldots,)$ *be defined thus*:

$$f_i(n) = \sum_{j=0}^{n-1} F_{i-1}(j),\ c_i = \frac{1}{p} \sum_{j=0}^{p-1} f_i(j),$$

$$F_i(n) = f_i(n) - c_i\,;\ F_0(n) = f(n). \qquad (4')$$

Then the series

$$\sum_0^\infty f(n)\,a(n)$$

is summable (C, k), *but not* $(C, k-1)$, *to the value*

$$\sum_{i=0}^{k-1} (-1)^i c_{i+1}\, \Delta^i a(0).^*$$

This follows from Theorem A′ and the analogue of Lemma 2 for sums(3) [Lemma 1] exactly as Theorem B follows from Theorem A and Lemma 2.

* $(-1)^i \Delta^i$ appearing here is the same as Δ^i in [3].

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ON TWO PROBLEMS RELATING TO LINER CONNECTED TOPOLOGICAL SPACES.

BY

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In this note I answer two questions of Dr R. Vaidyanathaswamy.

The questions are

(i) Suppose that a connected linearly ordered topological space S has the power of the continuum; does it follow that S has an everywhere dense enumerable subset?

(ii) A point P of a connected topological space S is said to be a *cut point* if its removal splits S into two (and only two) disjoint connected open topological spaces. If every point P of a connected topological space S is a *cut point* of S does it imply then that S is a linear space?

Both these questions are plausible enquiries and one would expect the answers to be in the affirmative. In this note it is shown that the answer is in the negative in each case.

The following example of a topological space S shows that the answer to the first query is in the negative

Let S consist of all points $P=(x, y)$, where $-\infty < x < \infty$, $0 \leqslant y \leqslant 1$. If $P=(x, y)$ and $P'=(x', y')$ and $x < x'$ then we say that $P < P'$. If $x = x'$ but $y < y'$ then too we say that $P < P'$; otherwise P is not less than P'. It is plain that between any two distinct P, Q of S there exists one and only one of the relations $P < Q$, $Q < P$. If P is any point of S and A and B are such that $A < P < B$ then all points Q for which $A < Q < B$ constitute a neighbourhood of P. If we picture the points as points in an infinite strip in the x, y plane then we may make the following remarks;

If $P=(x_0, y_0)$ and $0<y_0<1$ then every open interval on the line $x=x_0$ and containing x_0 is a neighbourhood of P; if $y_0=0$, then any neighbourhood of P necessarily contains a segment of the line $x=x_0$, beginning with $(x_0, 0)$ and a rectangle of which the line $x=x_0$ is the boundary on the right; similarly if $y_0=1$ then any neighbourhood of P necessarily contains a segment of the line $x=x_0$, and ending with $(x_0, 1)$ and a rectangle of which the line $x=x_0$ is the boundary on the left.

We see that the space so defined is a connected linear topological space S and that the points of S have the power of the continuum. It will now be shown that S contains no enumerable everywhere dense subset G. For, if $P=(x_0, y_0)$, $0<y_0<1$ and P is a limit point of G then the set G must have an infinity of points on the line $x=x_0$, and hence if G is everywhere dense then G has points on every line $x=x_0$ and hence is unenumerable and has the power of the continuum.*

The following example answers the second question. We picture S as consisting of points (x, y), where $1 \leqslant x^2+y^2<2$. If $P_0=(x_0, y_0)$, $1<x_0^2+y_0^2<2$ so that $P_0=(r_0 \cos \theta_0, r_0 \sin \theta_0)$, where $r_0=\sqrt{(x_0^2+y_0^2)}>1$, and δ is any positive number for which $1<r_0-\delta<r_0+\delta<2$, then $P_0(\delta)$ is defined to consist of all points P for which $P=(r \cos \theta_0, r \sin \theta_0)$, where $r_0-\delta<r<r_0+\delta$. In other words $P_0(\delta)$ is an open interval in the line joining the origin $O=(0, 0)$ to $P_0=(x_0, y_0)$, where the whole open interval belongs to S and has P_0 for its midpoint. We define $P_0(\delta)$ to be a neighbourhood of P_0 whenever δ satisfies the inequalities $1<r_0-\delta<r_0+\delta<2$. If $P_0=(r_0 \cos \theta_0, r_0 \sin \theta_0)$, $r_0=1$ then $P_0(\delta)$ denotes the set of points

* Plainly G has at least the power of the continuum and is a subset of S whose points have the power of the continuum. Hence by the Schroeder Bernstein theorem, G has precisely the power of the continuum.

(x, y) which belong to S and for which $(x-x_0)^2+(y-y_0)^2 < \delta$, and $P_0(\delta)$ is a neighbourhood of P_0 whenever $\delta > 0$. It is easy to verify that S is a connected topological space and that the removal of any point $P_0 = (r_0 \cos\theta_0, r_0 \sin\theta_0)$ splits S into two (and only two) disjoint open connected spaces of which the one which we call S_{P_0} consists (precisely) of the open interval containing all but the end points of the line joining $P_0 = (r_0 \cos\theta_0, r_0 \sin\theta_0)$ and $Q = (2 \cos\theta_0, 2 \sin\theta_0)$. We now show that S is not a linear space. Either of the following reasons is sufficient to imply the above conclusion :—

(1) If P_0, P_1 and P_2 are three points on three distinct rays proceeding from the origin, (i.e. θ_0, θ_1, θ_2 are all distinct) then each pair of the three subspaces S_{P_0}, S_{P_1}, S_{P_2} is disjoint.

(2) The set of all the points $P = (x, y)$, where $x^2+y^2 = 1$ is a subset of S and is a connected bit but contains no neighbourhood of any point P of S.

ON THE LARGEST PRIME DIVISORS OF NUMBERS

BY

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1. The object of this note is to give an answer to the QUERY: *Let $g(m)$ denote the largest prime divisor of m. In what range does $g(m)$ lie for all almost all values of $m \leqslant x$?*

More precisely, suppose that $h(x)$ and $H(x)$ are two functions of x; let $N(x) = N(h, H, x)$ denote the number of numbers $m \leqslant x$ for which $h(x) \leqslant g(m) \leqslant H(x)$. For what choices of $h(x)$ and $H(x)$ can we say that $N(x)/x \to 1$, as $x \to \infty$? One of the results proved in this paper is that $N(x)/x \to 1$ provided that $\log h(x) = o(\log x)$, $\log H(x) \sim \log(x)$; in a sense this is a best possible result. A more detailed account of the results of this paper is as follows.

Suppose that $0 < \theta < 1$ and $\phi(x, \theta)$ denotes the number of numbers $m \leqslant x$ for which $g(m) > x^\theta$. It is shown that

$$\lim_{x \to \infty} \frac{\phi(x, \theta)}{x}$$

exists and that if $\xi(\theta)$ denotes this limit, then $\xi(\theta)$ is a strictly decreasing continuous function in the interval $0 < \theta < 1$, $\xi(\theta) \to 0$ as $\theta \to 1$, and $\xi(\theta) \to 1$ as $\theta \to 0$. The actual value of $\xi(\theta)$ is evaluated as a sum of one or more integrals, from which it follows that $\xi(\theta)$ is an analytic function of θ in each of the intervals $1/(k+1) < \theta < 1/k$, $k = 1, 2, 3, \ldots$

2. The case of $\theta \geqslant \frac{1}{2}$ is easy to discuss. If p is prime $x^\theta < p \leqslant x$, $m \leqslant x$, $p \mid m$ then it follows that p is the largest prime divisor of m and that the number of numbers $m \leqslant x$ for which a particular such p is the largest prime divisor is equal to $[x/p]$, where, as usual, $[y]$ denotes the largest integer that does not exceed y. In what follows p always denotes primes. We now see that if $\theta \geqslant \frac{1}{2}$ then

$$\phi(x,\theta) = \sum_{x^\theta < p \leqslant x} [x/p]$$

$$= \sum_{x^\theta < p \leqslant x} \frac{x}{p} + O\left(\frac{x}{\log x}\right) \sim x \sum_{x^\theta < p \leqslant x} \frac{1}{p} \sim x \log \frac{1}{\theta}, \quad (1)$$

since by a well-known theorem*

$$\sum_{p \leqslant y} 1/p = \log \log y + B + o(1), \quad (2)$$

where B is a constant. Plainly then follows

THEOREM 1.

$$\lim_{x \to \infty} \frac{\phi(x,\theta)}{x} = \log \frac{1}{\theta} = \int_\theta^1 \frac{dt}{t}, \quad \tfrac{1}{2} \leqslant \theta < 1.$$

3. We next proceed to discuss the case $\frac{1}{3} \leqslant \theta < \frac{1}{2}$. If $\theta < \frac{1}{2}$ the formula (1) requires modification; if $\frac{1}{3} \leqslant \theta < \frac{1}{2}$ then we shall have

$$\phi(x,\theta) = \sum \left[\frac{x}{p}\right] - \sum \left[\frac{x}{pq}\right] = \Sigma_1 - \Sigma_2, \text{ say}, \quad (3)$$

where in Σ_1 the summation is over all primes $p > x^\theta$, and in Σ_2 it is over all pairs of primes p, q, where $x^\theta < p < q$, $pq \leqslant x$. We may remark that (1) is a particular case of (3) since if $\theta \geqslant \frac{1}{2}$ then Σ_2 contains no terms and is therefore equal to 0. The proof of (3) is fairly obvious since a number $m \leqslant x$ is divisible by utmost two distinct primes p, q each of which exceeds x^θ, $\theta \geqslant \frac{1}{3}$. From what has been said before we see that $\Sigma_1 \sim \log(1/\theta)$, and we proceed to show that from (2) and other well-known results, we can deduce (by straightforward reasoning) that

$$\sum_2 \sim x \int_\theta^{\frac{1}{2}} \frac{1}{t} \log \frac{1-t}{t}\, dt.$$

We can also write the above in the form

$$\sum_2 \sim x \int_\theta^{\frac{1}{2}} \frac{dt}{t} \int_t^{1-t} \frac{du}{u}.$$

To prove the above asymptotic formula for Σ_2 we observe, in the first place, that the number of terms in Σ_2

* E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 100-2.

is $o(x)$, since the numbers pq are a subset of numbers $\leqslant x$ which have two prime divisors only*. Hence

$$\sum_2 = x \sum \frac{1}{pq} + o(x), \tag{4}$$

where the summation is over all pairs of primes p, q for which $x^\theta < p < q, pq \leqslant x$. Now let $f(c, c') = f(c, c', x)$ denote the sum of the reciprocals of all primes p for which $x^c < p < x^{c'}$. If $\theta \leqslant c < c+\delta < \frac{1}{2}$, and $S(c, c+\delta)$ denotes the sum of the terms $\frac{1}{pq}$ which occur on the right hand side of (4) and for which $x^c < p \leqslant x^{c+\delta}$, then it is easily seen that

$$f(c, c+\delta) f(c, 1-c) \geqslant S(c, c+\delta) \geqslant f(c, c+\delta) f(c+\delta, 1-c-\delta).$$

From (2) we know that

$$\lim_{x\to\infty} f(c, c') = \log (c'/c), \; c' > c > 0,$$

and if δ is small and x large then both the bounds of $S(c, c+\delta)$ are nearly equal to† $\log \left(1+\frac{\delta}{c}\right) \log \frac{1-c}{c}$, which again is nearly equal to $\frac{\delta}{c} \log \frac{1-c}{c}$. Hence if n is a positive integer, $\delta = \frac{1}{n}\left(\frac{1}{2} - \theta\right)$, then plainly

$$\Sigma_2 = x \sum_{r=0}^{n-1} S\{\theta + r\delta, \theta + (r+1)\delta\} + o(x).$$

When n is large the right side of the above is nearly equal to

$$x \int_\theta^{\frac{1}{2}} \frac{1}{c} \log \frac{1-c}{c}\, dc = x \int_\theta^{\frac{1}{2}} \frac{dt}{t} \int_t^{1-t} \frac{du}{u}.$$

* E. Landu, *loc. cit.* pp. 203-13. Indeed from the prime number theorem we can deduce that the number of terms in Σ_2 is asymptotic to

$$\frac{x}{\log x} \int_\theta^{\frac{1}{2}} \frac{dt}{t(1-t)} = \frac{x}{2 \log x} \log \frac{1-\theta}{\theta}.$$

† 'Nearly equal' in the sense that the error is negligible in comparison with the value given, i.e. the ratio of the error to the value given is less than any arbitrarily fixed small positive number whenever the corresponding appropriate conditions are satisfied.

Hence we have

THEOREM 2. *If* $\frac{1}{3} \leqslant \theta < \frac{1}{2}$ *then*

$$\xi(\theta) = \lim_{x\to\infty} \frac{\phi(x, \theta)}{x} = \int_\theta^1 \frac{dt}{t} - \int_\theta^{\frac{1}{2}} \frac{dt}{t} \int_t^{1-t} \frac{du}{u}.$$

4. We shall now briefly indicate how the results of Theorems 1 and 2 can be extended for values of θ for which $0 < \theta \leqslant \frac{1}{3}$. If n is a positive integer and

$$1/(n+1) \leqslant \theta < 1/n,$$

then we see that

$$\phi(x, \theta) = \Sigma_1 - \Sigma_2 + \dots + (-1)^{n-1}\Sigma_n,$$

where

$$\Sigma_r = \Sigma\left[\frac{x}{p_1 p_2 \dots p_r}\right],$$

the summation being over all primes $p_1, p_2, \dots, p_r$ for which

$$x^\theta < p_1 < \dots < p_r, \quad p_1 p_2 \dots p_r \leqslant x.$$

As before we can prove that

$$\Sigma_r \sim x L_r(\theta), \; r = 1, 2, \dots, n$$

where $L_r(\theta)$ is equal to

$$\int_\theta^{1/r} \frac{dx_1}{x_1} \int_{x_1}^{(1-x_1)/(r-1)} \frac{dx_2}{x_2} \int_{x_2}^{(1-x_1-x_2)/(r-2)} \frac{dx_3}{x_3} \dots \int_{x_{r-1}}^{1-x_1-\dots x_{r-1}} \frac{dx_r}{x_r}. \tag{5}$$

It is easy to see that $L_r(\theta) \to 0$, as $\theta \to \frac{1}{r} - 0$ and that

$$\xi(\theta) = \sum_{r<[1/\theta]} (-1)^{r+1} L_r(\theta) \tag{6}$$

is a continuous function of θ in the interval $0 < \theta < 1$ and $\xi(\theta)$ is an analytic function of θ in each of the open intervals $\left(\frac{1}{n+1}, \frac{1}{n}\right)$, $n = 1, 2, \dots$ From the definition of $\xi(\theta)$ it is clear that it is a non-increasing function of θ and a closer scrutiny of (5) and (6) makes it clear that $\xi(\theta)$ is a strictly decreasing function of θ. The following elementary proof enables us to see at a glance that $\xi(\theta)$ is strictly decreasing.

We consider the set of numbers $m \leqslant x$, whose largest prime divisors are in the interval $(x^{\theta}, x^{\theta'})$ and such that the number of numbers in a suitable subset is comparable with x. We choose a sub-interval (a, b) in the interval (θ, θ') and such that

$$\theta \leqslant a < b \leqslant \theta', \; 1/(n+1) \leqslant a < b \leqslant 1/n,$$

where n is a positive integer. We see that the number of numbers $m \leqslant x$, which have in the interval (x^a, x^b) precisely n unequal prime divisors $p_1, p_2, \ldots, p_n$ is equal to

$$\sum \left[\frac{x}{p_1, p_2 \ldots p_n} \right],$$

where the summation is over all primes $p_1, \ldots, p_n$ for which

$$x^a < p_1 < p_2 < \ldots < p_n < x^b.$$

The numbers m considered here may have prime divisors less than x^a, but have no prime divisors $\geqslant x^b$. They have to be counted when we consider $\phi(x, \theta')$, but not when we consider $\phi(x, \theta)$. It is easy to verify that the above sum is asymptotically equal to*

$$\frac{x}{n!} \Big(\sum_{x^a < p < x^b} \frac{1}{p} \Big)^n \sim \frac{x}{n!} \left(\log \frac{b}{a} \right)^n,$$

i.e. a positive constant multiple of x. Hence $\xi(\theta) > \xi(\theta')$ whenever $0 < \theta < \theta' < 1$.

5. Finally we prove that

$$\xi(\theta) \rightarrow 1 \text{ as } \theta \rightarrow 0 \text{ and } \xi(\theta) \rightarrow 0 \text{ as } \theta \rightarrow 1.$$

The latter is obvious since if $\frac{1}{2} \leqslant \theta < 1$, then $\xi(\theta) \doteqdot \log (1/\theta)$. To show that $\xi(\theta) \rightarrow 1$ as $\theta \rightarrow 0$ we make use of an argument of Dr. S. S. Pillai communicated in a letter to one of us.† Let $v(x, y)$ and $V(x, y)$ denote respectively the number of numbers $m \leqslant x$ for which $g(m)$ is equal to

* The number of terms is $o(x)$ and those of the terms in the expansion of $\Big(\sum_{y^a < p < x^b} 1/p \Big)^n$ in which two or more primes p are equal, add up to $o(1)$.

† We may add that he has obtained precise results concerning the order of magnitude of $\xi(\theta)$ as $\theta \rightarrow 0$ and we understand that he will be publishing them shortly.

the largest prime that does not exceed y, and the number of numbers $m \leqslant x$ for which $g(m) \leqslant y$. Plainly

$$\phi(x, \theta) = x - V(x, x^{\theta}),$$

$$\lim_{x \to \infty} \frac{1}{x} V(x, x^{\theta}) = 1 - \xi(\theta).$$

Since $0 < \xi(\theta) < 1$ and $\xi(\theta)$ is monotonic in θ, it is evident that $\lambda(\theta) = 1 - \xi(\theta)$ tends to a limit c when $\theta \to 0$. Plainly $c \geqslant 0$. Our object is to prove that $c = 0$. We shall now prove that if $c > 0$ then there arises a contradiction. We have from the definition of $v(x, y)$ and $V(x, y)$ that

$$V(x, x^{2/n}) = V(x, x^{1/n}) + \sum_{x^{1/n} < p \leqslant x^{2/n}} v(x, p). \tag{7}$$

We now observe that

$$v(x, p) = V(x/p, p). \tag{8}$$

If n is large and x is correspondingly large then $V(x/p, p)$ is nearly equal to* cx/p and hence from (7) and (8) we deduce that

$$V(x, x^{2/n}) = V(x, x^{1/n}) + \sum_{x^{1/n} < p < x^{2/n}} V(x/p, p)$$

$$> V(x, x^{1/n}) + (c - \theta) \sum_{x^{1/n} < p < x^{2/n}} x/p$$

$$> V(x, x^{1/n}) + (c - \epsilon)(\log 2 - \epsilon)\, x,$$

which contradicts the assumption that if n is large and x correspondingly large, then each of the numbers $V(x, x^{2/n})$ and $V(x, x^{1/n})$ is nearly equal to cx.

6. Lastly we mention two general problems of the kind considered in this note. Let $p_1, p_2, \ldots, p_k$ be any k primes, $p_1 < p_2 < \ldots < p_k \leqslant x$ and let $T(p_1, p_2, \ldots, p_k, x)$ denote the number of numbers $m \leqslant x$, where m is divisible

* If we write $x/p = y$, $p = y^{\theta}$ then $2/(n-2) \leqslant \theta < 1/(n-1)$ since $x^{1/n} < p \leqslant x^{2/n}$ and therefore $p^{(n/2)-1} \leqslant x/p < p^{n-1}$. Now $\frac{1}{y} V(y, y^{\theta})$ is an increasing function of θ and hence if x and therefore y too be large, then $V(y, y^{\theta}) \geqslant V(y, y^{2/(n-2)}) > y \left\{ \lambda\left(\frac{2}{n-2}\right) - \frac{\epsilon}{2} \right\}$, where ϵ is any previously assigned positive constant. If n is large then $\lambda\left\{2/(n-2)\right\} - \epsilon/2 > c - \epsilon$, since $\lambda(t) \to c$ as $t \to 0$,

by at least one of the primes $p_1, \ldots, p_k$. Let $\psi(\delta)$ denote the greatest lower bound of $T(p_1, \ldots, p_k, x)/x$ for all k, for all sets $p_1, \ldots, p_k$ and for all x, where $\frac{1}{p} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} \geqslant \delta$.

The problems are:

(i) Does $\psi(\delta) \to 1$ as $\delta \to \infty$?

(ii) Is $\psi(\delta) > 0$ whenever $\delta > 0$?

It is, of course, easy to see that $\psi(\delta) \to 0$ as $\delta \to 0$ since

$$T(p_1, \ldots, p_x, x) \leqslant \sum_{t=1}^{k} [x/p_t] \leqslant x \sum_{t=1}^{k} 1/p_t$$

SOME PROPERTIES OF DISTANCE FUNCTIONS.

BY

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1. Introduction. The object of this paper is to discuss some properties of distance or metric functions. Given any abstract set of elements R, a non-negative real valued function $d(x, y)$ defined for every pair of elements (x, y) of R is called a *metric* or a *distance function* if the following conditions hold:—

(i) $d(x, x) = 0$;

(ii) $d(x, y) > 0$ if $x \neq y$;

(iii) $d(x, y) = d(y, x)$;

(iv) $d(x, y) \leqslant d(x, z) + d(z, y)$

for any three elements x, y, z of R. If $d(x, y)$ satisfies the above conditions except (ii), it is called a *quasi-metric* or a *quasi-distance function.* This metric defines a topology on R and R is then called a metric space with the metric $d(x, y)$. In general, different metrics define different topologies. In this paper we discuss the properties of non-negative real-valued functions $\phi(z)$, $z \geqslant 0$, such that $\phi[d(x, y)]$ still remains a metric and consider the conditions under which the topology also remains unaltered. We discuss also some application to vector-metric spaces.

2. Theorem I. *Let $\phi(x)$ be a function of a real variable defined for $z \geqslant 0$. In order that for every metric $d(x, y)$, the function $\phi[d(x, y)]$ should also be a metric, it is necessary and sufficient that (i) $\phi(0) = 0$; (ii) $\phi(z) > 0$ for $z > 0$ and (iii) $\phi(z_3) \leqslant \phi(z_1) + \phi(z_2)$ for $|z_1 - z_2| \leqslant z_3 \leqslant z_1 + z_2$.*

Proof. Necessity follows by taking $d(x, y) = |x - y|$ for the real number space. To prove sufficiency, (i) and (ii) of §1 for $\phi[d(x, y)]$ are consequences of (i) and (ii) in the theorem; (iii) of §1 is obvious and (iv) is a consequence of (iii) of the theorem.

We shall call a function satisfying (i), (ii) and (iii) of Theorem 1 a *metric transforming function.* It may be noted that *any such function transforms a quasi-metric into a quasi-metric.*

2.1. THEOREM 2. *A non-constant increasing concave function* $\phi(z)$ *with* $\phi(0) = 0$ *is a metric transforming function.*

PROOF. If $0 \leqslant z_1 < z_2 < z_3$, concavity gives

$$\phi(z_2) \geqslant \phi(z_1) \frac{z_3 - z_2}{z_3 - z_1} + \phi(z_3) \frac{z_2 - z_1}{z_3 - z_1}. \tag{1}$$

Hence putting $z_1 = 0$ we get for $0 < z_2 < z_3$,

$$\phi(z_2)/z_2 \geqslant \phi(z_3)/z_3. \tag{2}$$

Since $\phi(0) = 0$ and $\phi(z)$ increases, it is enough to show that (iii) of Theorem 1 holds with $z_3 = z_1 + z_2$. From (1) we get

$$\begin{aligned} \phi(z_1+z_2) &\leqslant \phi(z_2) \frac{z_2}{z_2 - z_1} - \phi(z_1) \frac{z_1}{z_2 - z_1} \\ &= \phi(z_1) + \phi(z_2) + \frac{z_1 z_2}{z_2 - z_1} \left[\frac{\phi(z_2)}{z_2} - \frac{\phi(z_1)}{z_1} \right] \\ &\leqslant \phi(z_1) + \phi(z_2) \end{aligned} \tag{3}$$

by using (2), since $z_1 < z_2$. This proves the theorem.

2.2. By using the corresponding property of a convex function we can prove similarly that a convex function $\phi(z)$, vanishing at $z = 0$ satisfies $\phi(z_1+z_2) > \phi(z_1) + \phi(z_2)$ for properly chosen values of z_1 and z_2 and so cannot be a metric transforming function.

2.3. EXAMPLES. (1) The function $d(x,y) = |x^2 - y^2|$ is a quasi-distance in the real number space.

(2) The functions $\frac{z}{1+z}$, z^k $(0 < k \leqslant 1)$, $\log(1+x)$, $\phi(x) = \sin x$, $0 \leqslant x \leqslant \pi/2$ and $= 1$ for $x > \pi/2$ and $1 - e^{-x}$ are all metric transforming functions.

(3) If $f(x)$ and $\phi(x)$ both satisfy the hypothesis of Theorem 2, $f\{\phi(x)\}$ is a metric transforming function. This enables one to construct other examples of such functions.

3. Suppose $d(x, y)$ satisfies (i), (ii) and (iii) or (i) and (iii) of §1 but not (iv). Suppose further that for some real z (obviously $\geqslant 1$), $\{d(x,y)\}^{1/z}$ satisfies (iv). Then the same is true for any $z' > z$. Hence the lower bound ρ of such numbers exists for which also (by proceeding to the limit in (iv) of §1) (iv) of §1 holds so that $\{d(x,y)\}^{1/\rho}$ is a metric or a quasi-metric. We shall call ρ the *distance-index* of $d(x,y)$. If no finite ρ exists we shall say that $d(x, y)$ has the distance-index infinity. For instance in the real number space, $d(x, y) = |x-y|^m$ has the distance index m when $m > 1$. The function $d(x, y) = e^{|x-y|}$ has the distance-index infinity since if a, b are sufficiently large $e^{a+b} > e^a + e^b$.

4. VECTOR-METRIC SPACES. We shall mean by a vector α a sequence $\{a_n\}$ of real numbers. The class of vectors is partially ordered by the relation $\alpha \leqslant \beta$ if this means that $a_n \leqslant b_n$ [where $\alpha = \{a_n\}$, $\beta = \{b_n\}$] for all n. If θ denotes the zero vector $\{0\}$, we shall say that α is non-negative if $\alpha \geqslant \theta$, i.e. $\theta \leqslant \alpha$.

Consider now a space R. Let a vector $V(x, y)$ be defined for each pair x, y of R. We shall call $V(x, y)$ a *vector-metric* and R *a vector-metric space* if the following conditions hold:

(i) $V(x, y) \geqslant \theta$ and $= \theta$ if and only if $x = y$;

(ii) $V(x,y) = V(y, x)$;

(iii) $V(x,y) \leqslant V(x, z) + V(z, y)$

for any three elements x, y, z of R. We have now the following

THEOREM 3. *If* $V(x, y) = \{d_n(x, y)\}$ *be a vector-metric then each* $d_n(x,y)$ *is a quasi-metric function; conversely, if each* $d_n(x, y)$ *is a quasi-metric and the relations* $d_n(x,y) = 0$ *for all* n *imply* $x = y$, *then* $V(x,y) = \{d_n(x,y)\}$ *is a vector-metric.*

The proof of this theorem is an immediate consequence of the definitions concerned. In particular if at least one

$d_n(x, y)$ is a metric function and the other quasi-metric then $V(x, y)$ is a vector-metric.

4.1. As an example consider a sequence $\{R_n\}$ of metric spaces, R_n having $d_n(x, y)$ for the metric. Let R be the product space of the sequence $\{R_n\}$, i.e. R consists of all elements $\alpha = (x_1, x_2, \ldots)$, where $x_n \epsilon R_n$ for $n = 1, 2, \ldots$ Then $V(\alpha, \beta) = \{d_n(x_n, y_n)\}$, where $\alpha = \{x_n\}$ and $\beta = \{y_n\}$ is a vector-metric, since (i), (ii) and (iii) are easily seen to be satisfied by using the properties of a metric.

4.2. If $\phi(x)$ is a real-valued function defined for real x, we mean by the vector function $\phi(\alpha)$, $\alpha = \{a_n\}$, the vector $\{\phi(a_n)\}$. The following is an easy consequence of the definition of a metric transforming function and the previous theorem.

THEOREM 4. *If $\phi(z)$ is a metric transforming function and $V(x, y)$ is a vector-metric then so is $\phi\{V(x, y)\}$.*

4.3. Let R be a vector-metric space with vector-metric $V(x, y)$. This vector-metric induces a topology in R in the usual manner.* The question now arises whether this topology can be *induced by an ordinary metric*. The answer is in the affirmative as shown by

THEOREM 5. *Let $V(x, y)$ be a vector-metric in R. Then*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)}$$

gives a metric which induces the same topology on R as the vector-metric $V(x, y) = \{d_n(x, y)\}$.

PROOF. That $d(x, y)$ is a metric follows from the properties of $d(x, y)$ and the fact that $\phi(z) = \frac{z}{1+z}$ is a

* If (x_p) is any sequence of elements of R we say that $x_p \to x$ as $p \to \infty$ in the vector-metric $V(x, y)$ if $V(x_p, x) \to \theta$ which is interpreted as $d_n(x_p, x) \to 0$ as $p \to \infty$ for each n. Now $x \epsilon \bar{A}$ where $A \subset R$ if there is a sequence $(x_p) \subset A$ such that $x_p \to x$. This gives the topology induced by $V(x, y)$ in R.

metric transforming function. Since $V(x_p, x) \to \theta$ as $p \to \infty$ is equivalent to $d_n(x_p, x) \to 0$ as $p \to \infty$ for each n, it follows that each of these is equivalent to $d(x_p, x) \to 0$ as $p \to \infty$. Hence $V(x, y)$ and $d(x, y)$ give the same topology.

4.4. We can define the distance-index of a non-negative vector $V(x, y)$ as the lower bound of numbers λ such that $\{ V(x, y) \}^{1/\lambda}$ is a vector-metric, it being understood that $V(x, y)$ satisfies (i) and (ii) of §4. In this connection we have the following

THEOREM 6. *The distance-index of a non-negative vector $V(x, y)$ is the upper bound of the distance-indices of the components $\{ d_n(x, y) \}$ of $V(x, y)$.*

PROOF. Let ρ be the distance-index of $V(x, y)$ and λ the upper bound of those of $d_n(x, y)$. If $\rho' > \rho$, then $\{ V(x, y) \}^{1/\rho'}$ is a vector-metric; hence by Theorem 3, the distance-index of each $d_n(x, y)$ is $\leqslant \rho'$. Hence $\lambda \leqslant \rho'$. Since $\rho' > \rho$, but otherwise arbitrary $\lambda \leqslant \rho$. On the other hand, by the definition of distance-index of $d_n(x, y)$ we see that $\{ d_n (x, y) \}^{1/\lambda}$ is a quasi-distance function for each n and hence $\{ V(x, y) \}^{1/\lambda}$ is a vector-metric. So $\rho \leqslant \lambda$. So $\rho = \lambda$.

4.5. Another result in this connection is

THEOREM 7. *Let $V(x, y)$ be a non-negative vector defined for x, y on R and satisfying (i) and (ii) of §4. Let its distance-index be ρ. Let P be a constant non-negative vector. Then the distance-index of the inner product of P and $V(x, y)$ does not exceed ρ provided the inner product exists as a finite number for each pair (x, y).*

PROOF. Let $V(x, y) = \{ d_n (x, y) \}$ and $P = \{ a_n \}$. Let $d(x, y) = \Sigma a_n d_n (x, y)$ which is finite by hypothesis. By Theorem 6,

$$\{ d(x, y) \}^{1/\rho} = \{ \Sigma a_n d_n (x, y) \}^{1/\rho}$$
$$\leqslant \{ \Sigma a_n d_n (x, z) \}^{1/\rho} + \{ \Sigma a_n d_n (z, y) \}^{1/\rho}$$

by Minskowski's inequalities since $\rho \geqslant 1$.

Hence the result.

5. We conclude with a few remarks on the metric transforming functions. Theorem 2 shows that an increasing concave function $\phi(z)$ with $\phi(0) = 0$ is such a function. But neither of these conditions is necessary. The following examples illustrate the various possibilities :—

(1) $\phi(x) = x$ in $(0, 2)$; $\phi(x) = 4 - x$ in $(2, 3)$ and $\phi(x) = 1$ in $(3, \infty)$ is verified to be a metric transforming function and it decreases in $(2, 3)$ and increases in $(0, 2)$. Hence it is not monotonic. This is not concave in an interval including $x = 3$.

(2) $\phi(x) = \sqrt{2x - x^2}$ in $(0, 1)$ and $\phi(x) = x$ in $(1, \infty)$ is again metric transforming and increasing in $(0, \infty)$ but it is not concave in any interval including $x = 1$.

(3) By joining together graphs of concave functions in a suitable manner we can get a metric transforming function having any number of points of inflection and at the same time monotonic increasing.

5.1. It is interesting to find out *when the two metrics* $d(x, y)$ and $\phi[d(x, y)]$ *are equivalent topologically*. Evidently if *the relation* $z_n \to +0$ *is equivalent to* $\phi(z_n) \to +0$, *then the two metrics are equivalent. In particular if* $\phi(z)$ *is continuous and increasing in the neighbourhood of* $z = +0$, the two metrics are equivalent but it looks as if no simple necessary and sufficient condition on $\phi(z)$ can be found to ensure equivalence.

My thanks are due to Dr. R. Vaidyanathaswami for suggesting this problem and for valuable criticisms in the preparation of this paper. My thanks are also due to the referee for his assistance in improving the presentation of the paper.

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ON THE PRODUCT OF THE ELEMENTS IN A FINITE ABELIAN GROUP.

BY

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1. The object of this paper is to prove some theorems regarding the product* of elements in a finite abelian group. The main theorem is

THEOREM 1. *The product of the elements of a finite abelian group G is always the identity unless there is, in G, a single element A of order 2 when the product is equal to A.*

Some applications of this theorem are shown in the sequel. Let $R(k)$ denote the group of prime residue classes mod k. Let $k = 2^{\alpha} p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ and $2^{\beta_1}, 2^{\beta_2}, \ldots$ the various powers of 2 dividing $p_1 - 1, p_2 - 1, \ldots$ respectively. Let μ_1 be the highest of the numbers $1, \alpha - 2, \beta_1\ \beta_2, \ldots$ and $\mu_2 \leqslant \mu_1$ the next highest. A typical theorem concerning $R(k)$ is

THEOREM 2. *Let $n = 2^{\nu} m$, m being odd, be a divisor of the highest† invariant λ_1 of $R(k)$ and let ξ run through all the n-ic‡ residues* mod k. *Then*

$$(i)\quad \Pi\, \xi \equiv 1 \pmod{k}$$

if $\nu = \mu_1$ *or* $\nu < \mu_2$;

$$(ii)\quad \Pi\, \xi \equiv A \pmod{k}$$

if $\mu_1 > \nu \geqslant \mu_2$ *and A is the n-ic residue of order* 2.

* The group operation is taken to be multiplication.

† The highest invariant λ_1 of $R(k)$ is called the *exponent* by A. A. Albert. See his '*Modern Higher Algebra*', p. 130. The exponent of a finite abelian group is the order of an element of maximum order.

‡ a is said be a n-ic residue mod k if $(a, k) = 1$ and

$$a \equiv x_n \pmod{x}$$

for some x.

It can at once be recognised that Theorem 2 is a generalization of Wilson's theorem*.

2. PROOF OF THEOREM 1.

We require the following

LEMMA. *The product of elements of G is equal to the product of elements of order* 2.

PROOF. Every element g which is not the identity has a unique inverse g^{-1}. g and g^{-1} are different if g is not of order 2. Since the group is commutative we may multiply each element by its inverse.

To prove Theorem 1 it is enough if we consider the elements of order 2 in G. Let $a_1, a_2, \ldots a_{t-1}$ $(t = 2^{\mu})$ be the elements in G of order 2. Let a_{t-1} be one of the elements. Since the elements of order 2 together with the identity form a group we see that

$$a_r\, a_s = a_{t-1}.$$

Hence the product of all the elements of order 2 is $a_{t-1}^{2^{\mu}-1}$. If $\mu > 1$ then $a_{t-1}^{2^{\mu}-1}$ is the identity otherwise it is a_{t-1}.

3. Let G be an abelian group of order $n = p_1^{a_1} \ldots p_r^{a_r}$. It is known† that G is the direct product of all its Sylow subgroups. Also an abelian group of order $p_i^{a_i}$ can be represented as the direct product of independent cyclic subgroups $c_{i1}, c_{i2}, \ldots$ of orders $p_i^{a_{i1}}, p_i^{a_{i2}}, \ldots$ respectively where $a_{i1} \geqslant a_{i2} \geqslant \ldots$. The abelian group of order n can thus be represented as the direct product of cyclic groups

$$\begin{array}{l} c_{11}, c_{12}, \ldots \\ c_{21}, c_{22}, \ldots \\ \ldots \\ c_{n1}, c_{n2}, \ldots \end{array}$$

Taking the direct product of cyclic groups in each column and using the fact that the direct product of cyclic groups

* Theorem 2 is substantially equivalent to Theorems 1 and 2 of P. Kesavan Menon, *Jour. Ind. Math. Soc.*, (2) 9 (1945), 79.

† Everything regarding abelian groups is contained in G. A. Miller, *Theory and applications of finite groups*, p. 89-94.

of coprime orders is cyclic, we arrive at the fundamental

THEOREM. *A finite abelian group is the direct product of independent cyclic groups of orders* $\lambda_1, \lambda_2, \ldots \lambda_r$ *such that* $\lambda_i \mid \lambda_{i-1}$ $(i = 2, 3, \ldots)$.

It is evident from the above method that an abelian group may be represented as the direct product of cyclic groups in many ways. The numbers $\lambda_1, \lambda_2 \ldots \lambda_r$ which are fixed for a given abelian group may be called *canonical invariants*. The generators $S_1, S_2, \ldots S_r$ of the independent cyclic groups form a *canonical basis* so that every element A of G may be uniquely written in the form

$$A = S_1^{h_1} S_2^{h_2} \ldots S_r^{h_r} \quad (0 \leqslant h_i < \lambda_i).$$

If $k_1, k_2, \ldots$ be the orders of the cyclic groups in any resolution, it is evident that

$\lambda_1 = \text{l.c.m.} [k_1, k_2, \ldots]$

$\lambda_2 = \text{l.c.m.}$ [g.c.d's of $k_1, k_2, \ldots$ taken 2 at a time]

...

and so on.

Let $G^{(n)}$ be the subgroup of elements A of G for which the equation

$$x^n = A$$

is soluble with x in G. It is easy to prove that the canonical invariants of $G^{(n)}$ are $\lambda_1/n_1, \lambda_2/n_2, \ldots$ where $n_i = (\lambda_i, n)$. Assuming that n is a divisor of λ_1—which implies no loss of generality—we prove

THEOREM 3. *The product of elements of* $G^{(n)}$ *is*

(*i*) *the identity if* λ_1/n_1 *is odd or if* λ_2/n_2 *is even*

(*ii*) *the unique element of order 2 in* $G^{(n)}$, *in all other cases*.

PROOF :—

The product of elements of $G^{(n)}$ is, by Theorem 1, the elements of order 2 in $G^{(n)}$. If λ_1/n_1 is odd then $G^{(n)}$ is of odd order and there is no element of order 2. If λ_2/n_2 is even, *a fortiori* λ_1/n_1 is, then there are at least 3 elements

of order 2. If λ_1/n_1 is even and λ_2/n_2 is odd there is only one element of order 2 in $G^{(n)}$. Applying Theorem 1 to all these cases the theorem is proved.

4. All these theorems on finite abelian groups hold good for $R(k)$, the group of prime residue classes mod k. The residue classes prime to k form an abelian group of order $\phi(k)$. It is known* that if p is an odd prime $R(p^\alpha)$ is cyclic and if $p = 2$ then $R(2^\alpha)$ is cyclic when $\alpha \leqslant 2$ and non-cyclic when $\alpha \geqslant 3$. In the latter case it is the direct product of two cyclic groups of orders 2 and $2^{\alpha-2}$. Also if $k = 2^\alpha p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ it is known that $R(k)$ is the direct product of $R(2^\alpha)$, $R(p_1^{\alpha_1})$,....It can be shown that if $\lambda_1, \lambda_2, \ldots$ are the canonical invariants of $R(k)$ then

$$\lambda_1 = \text{l.c.m. } [\phi(p_1^{\alpha_1}), \ldots, \phi(p_r^{\alpha_r})]$$

$$\lambda_2 = \text{l.c.m. } [\text{g.c.d's of } 2, \phi(p_1^{\alpha_1}), \ldots \text{ taken 2 at a time}]$$

if $\alpha < 3$ and

$$\lambda_1 = \text{l.c.m. } [2_1^{\alpha-2}, \phi(p_1^{\alpha_1}), \ldots]$$

$$\lambda_2 = \text{l.c.m. } [\text{g.c.d's of } 2, 2^{\alpha-2}, \phi(p_1^{\alpha_1}) \ldots \text{ taken 2 at a time}]$$

if $\alpha \geqslant 3$. The totient function $\phi(k)$ is even if $k > 2$ so that $R(k)$ is never of odd order and it always contains elements of order 2. It contains one element of order 2 when cyclic, otherwise at least 3. The analogue of Theorem 1 therefore is

THEOREM 4. *The product of elements of* $R(k)$ *is* ∓ 1 (mod k) *according as* $R(k)$ *is or is not cyclic.*

This is the familiar theorem of Gauss* in rational arithmetic. Theorem 2 of § 1 is the analogue of Theorem 3; for since the value of the product depends only on the residues of λ_1/n_1 and λ_2/n_2 (mod 2) it is enough if we consider the two numbers μ_1 and μ_2 in relation to ν, the index of the highest power of 2 dividing n.

* *See* Dirichlet-Dedekind '*Vorlesungen über Zahlentheorie*' or E. Hecke '*Theorie der algebrai schen zahlen*', p. 55 et. seq.

5. It is interesting to study the product of n-ic non-residues. This is easy in view of Theorems 2 and 4. We shall be content with stating these results.

THEOREM 5. *Let A, ν, μ_1, μ_2 have the same meanings as in Theorem 2 and let ξ run through all n-ic non-residues* mod k. *Then*

(*i*) *If $R(k)$ is cyclic*

$$\Pi\,\xi \equiv \pm 1 \pmod{k} \text{ according as } \nu = 0 \text{ or } 1.$$

(*ii*) *If $R(k)$ is non-cyclic*

$$\Pi(\xi) \equiv 1 \pmod{k} \qquad \nu > \mu_1 \text{ or } \nu < \mu_2$$

$$\Pi(\xi) \equiv A \pmod{k} \qquad \mu_1 > \nu \geqslant \mu_2.$$

THEOREM 6. *If m and n are two divisors of λ_1 containing the same power of 2 as a block factor then the product of the mth power residues is equal* (mod k) *to the product of the nth power residues.*

THEOREM 7. *If k is divisible by* 16 *or contains at least one prime factor of the form* $4n+1$, *the product* mod k *of all quadratic residues is equal to the product of non-residues.**

* Compare Hardy and Wright, *Introduction to the theory of Numbers*, p. 85 Theorem 113.

EXTENSIONS OF PARTIALLY ORDERED SETS*—I: GENERAL THEORY.

BY

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1. Introduction.

1.1. The partially ordered set.†

1.11. Def. An abstract set, L, is a *partially ordered set*, (or *ordered set*), if there is an associated binary relation "$<$", (read "less than"), connecting some pairs of elements of L, which is further *reflexive* and *transitive*‡ [that is : $a < a$ for all a of L; and $a < b$, $b < c$ imply $a < c$ for all a, b, c from L].

Though it is generally true that $a < b$, $b < a$ together imply $a = b$ (or a, b are identical), we do not include this in the definition. It may happen that there is an *equivalence relation*, $\approx$, [that is, an ordering relation which is also symmetric so that $a \approx b$ implies $b \approx a$] on L, defined in some natural manner for the objects forming the set L, which is not the same as the relation between two elements a, b defined by the statement "$a < b$ and $b < a$". Then the two relations can be combined to give a new equivalence relation. We shall have occassion to use such a construction later in dealing with the family of extensions. Now we can take note of a natural form of equivalence relation among the ordered sets themselves; namely the 'isomorphism'.

1.12. Def. Two ordered sets L; L' are *isomorphic*, in symbols $L \approx L'$, if there is a one-one reversible transformation, $i : x \rightarrow x'$, which maps each element x of L on an

* A thesis on this subject formed the basis for a joint award of the Ramanujan Memorial Prize, (1945) by the Madras University.

† For a more detailed treatment see MacNeille's paper on 'Partially ordered sets', and other references quoted in the bibliography at the end of this section. This paper of MacNeille is quoted in the sequel as 'M'.

‡ References for the Calculus of Relations are also given at the end of this section.

element x', $= i(x)$, of L' in such a way that $x < y$ in L if, and only if, $i(x) < i(y)$ in L'. The transformation i itself is called an *isomorphism of L on L'*.

Clearly isomorphic ordered sets have the same order structure and differ only in the naming of the elements. Structurally they are indistinguishable and so may be identified.

1.13. DEF. A *univocal order operation* (or, shortly, *operation*) θ is a method of associating to each subset A, [subject possibly to some restriction on its cardinal number], of *any* ordered set L, at most one element a of L; the element a being completely specified by its relations of order with the elements of L and those of A. When such an element a exists, it is denoted by $\theta_L(A)$, so that $a = \theta_L(A)$.

It is to be noted that the existence of $\theta(A)$ is not guaranteed by the definition, but only its uniqueness if it exists. [For example (lattice) *addition* and *multiplication* are defined for all non-null subsets of any ordered set; *finite addition* or *multiplication* is defined for non-null finite subsets only. The finite sum $+_L(a_1, a_2, \ldots, a_n)$ is usually written in the more familiar form $(a_1+a_2+\ldots+a_n)_L$.]

1.14. DEF. The ordered set L is *closed under the operation* θ, if $\theta_L(A)$ exists in L for every subset A of L [whenever A satisfies any restrictions implicit in the definition of θ]. For example a finite additive system is closed for finite addition, and so contains a sum for every non-null finite subset.

1.2. MACNEILLE'S THEORY OF CANONICAL EXTENSIONS. Certain special types of ordered sets like the modular and distributive lattice, and the Boolean algebra are particularly important; they possess special properties and are useful in characterising the structure of other algebraic systems also. These special ordered sets are specified by their closure under certain combinations of operations. In considering more general ordered sets

it is worthwhile to see if they can be in any way related to such special ordered sets One such natural relation would be that of being a subset of such a special ordered set, with the ordering relation the same as in the larger set. This leads us to the question of finding when an ordered set K, not closed under a set ϕ, of operations, would be isomorphic with [and so have the same structure as] a subset K', of another ordered set L' closed under ϕ. When there is such an isomorphism L' may be called *an extension to* ϕ *of* K *with* K' *as the* (isomorphic) *image of* K.

In the paper '*M*', MacNeille defines a form of partial order among such extensions and defines a 'canonical extension' which, if it exists, is a minimal under the ordering relation and is unique as such, to within isomorphic ordered sets.

This theory of extension is developed by MacNeille in two main stages. At first is developed a general abstract theory of extensions of arbitrary ordered sets for arbitrarily assigned sets of operations. This is followed up by actual constructions of canonical extensions of various types of ordered sets to various combinations of the operations of addition, multiplication or complementation. It is these constructions which form the important contributions of the paper. On the other hand the general theory seems to have developed intuitively along lines suggested by these later constructions, and does not appear to have been worked out rigorously as an independent logical theory. As a matter of fact, it is the object of this paper is to show that the general theory of canonical extensions contains serious flaws, and has to be revised considerably if it is to fulfill the intentions of the paper '*M*'

Apart from some verbal inaccuracies which can be easily adjusted, the main flaws found in the above theory are in (1) *the fundamental lemma* which states that the

identity is the only isomorphism of a canonical extension into itself which leaves invariant the elements of the image of K [M : § 9.9] and (2) the theorems relating to combination and resolution of canonical extensions [M: §§ 9.11, 9.13]. *These results cannot be proved from the definition of canonical extension given.* To prove this the only way is to construct contra-examples. Examples to establish the above, and also some further contentions regarding the theory in 'M' are constructed in § 3, after a preliminary section (§ 2), which contains all the necessary definitions, as also results from 'M' which come in for critical examination. In a final section I develop a modified theory of canonical extensions preserving as much as possible the developement in 'M', but free from the contradictions which arise there.

In a second part of this paper I shall consider explicit constructions of canonical extensions according to the modified theory, and also discuss cases of non-existence of canonical extensions.

I wish to offer here my thanks to Dr. R. Vaidyanathaswamy, for kindly going through this paper and suggesting corrections and alterations.

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2. Basic definitions and results from '*M*'.

2.1. Subsystem, supersystem, and superset. Let L_i be any ordered set and ϕ a set of univocal order operations; then we have the definitions:

2.11. Def. A subset, K_i, of L_i is *a subsystem of* L_i *for* ϕ provided, whenever $\theta_{L_i}(A)$ exists in L_i for $A \subset K_i$ and $\theta \epsilon \phi$, $\theta_{K_i}(A)$ also exists in K_i and equals $\theta_{L_i}(A)$.

2.12. L_i is a *supersystem for* ϕ *of* a subset, K_i, if, whenever $\theta_{K_i}(A)$ exists in K_i for $A \subset K_i$ and $\theta \epsilon \phi$, $\theta_{L_i}(A)$ also exists in L_i and equals $\theta_{K_i}(A)$.

In '*M*' there is a confusion between these two relations between L_i and K_i (vide 3.2 below). But these two relations are independant of each other as we see from the following example.

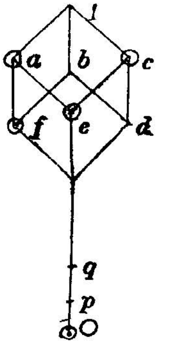

Fig. 1. K.

Example 1. Consider the set L_i; [Fig 1, gives the lattice diagram of L_i; the convention being that one element is $<$ another if in the diagram there is an ascending path from the point representing the first element to the point representing the second; thus $p < c$, $d \nless a$ in L_i]. The elements $o, p, q, r\ldots$form an infinite ascending chain with no highest element. Thus in L_i the products $d.e$, $d.f$, $f.e$ do not exist. The encircled points represent the elements of a subset K_i of L_i. We observe that $(f+e)_{K_i} = a$, $(a+c)_{K_i}$, $(f+c)_{K_i}$ do not exist, and $(f.e)_{K_i} = (f.c)_{K_i} = 0$, while $(f.e)_{L_i}$, $(f.c)_{L_i}$ do not exist, and $(f+e)_{L_i} = a$, $(f+c)_{L_i} = 1$. For all other pairs of elements of K_i, the sum or product is the same in L_i and in K_i. Hence L_i is a supersystem of K_i for finite addition but not for finite multiplication; while K_i is a subsystem for finite multiplication but not for finite addition.

Let L_i, K be two ordered sets, ϕ a family of operation; then

2.13. DEF. L_i is *a superset to* ϕ *of* K, provided (i) there is an isomorphism, $i: K \to K_i$, of K on a subset K_i of L_i, and (ii) L_i is closed under ϕ.

We call K_i *the (isomorphic) image of* K in L_i. This is one part of the definition of extension given in '*M*', but as some results are stated to be true for these supersets also, it is convenient to have a definition for this. The ordering relation defined in '*M*' applies to supersets as well as to extensions; the relation is:

2.14. DEF. $L_i \subset L_j$, where L_i, L_j are supersets to ϕ of K with K_i, K_j as images of K, if there is an isomorphism h_{i_j} of L_i on a subset of L_j such that $h_{i_j}(K_i) = K_j$. [Compare *M*: §9.2].

Clearly h_{i_j} is also an isomorphism of K_i on K_j; also $\subset$ is reflexive and transitive. It is thus an ordering relation for the family of supersets to ϕ of K. We shall see later (vide 3.4) that the simultaneous relations $L_i \subset L_j$, $L_j \subset L_i$ do not together imply nor are implied by the isomorphism $L_i \approx L_j$. So, as stated in the introduction, we consider a stronger equivalence relation combining the two above; namely,

2.15. DEF. L_i, L_j are super-isomorphic, and we set $L_i \approxeq L_j$, if there is an isomorphism, $h_{ij}: L_i \to L_j$, of L_i *on* L_j such that $h_{ij}(K_i) = K_j$.

Thus the super-isomorphism, h_{ij} between two supersets to ϕ of K is an isomorphism between L_i, L_j which is also an isomorphism between K_i, K_j.

2.2. THE EXTENSION, MINIMAL SUPERSET AND MINIMAL EXTENSION.

An extension, in the sense of '*M*', is a special type of superset and may be defined as follows:

2.21. DEF. L_i is an extension to ϕ of K, if it is a superset to ϕ of K and the image K_i of K in L_i is a subsystem of L_i for ϕ [compare *M* § 9.1].

The relations $\subset, \cong, \approx$ are defined for extensions, as they are defined among all supersets.

2.22. DEF. An extension, or superset, is said to be a *minimal* if it is $\subset$ every other extension, or superset.

As $L_i \subset L_j, L_j \subset L_i$ together neither imply nor are implied by $L_i \approx L_j$, two minimals may exist which are not isomorphic, and so not super-isomorphic either. The canonical extension is defined in '*M*' so as to be a minimal, unique to within isomorphisms.

2.3. REQUIRED ELEMENTS, REQUIRED RELATIONS AND THE CANONICAL EXTENSIONS OF '*M*'.

The canonical extension is defined in '*M*' using the auxiliary concepts of required elements and relations. These definitions are quoted below.

2.31. DEF. "An *element* of L_i, an extention to ϕ of K, is *required* by ϕ and K, if it can be represented by a finite succession of operations of ϕ on subsets of K_i" [*M* : § 9.5].

Such a required element has, thus, one or more representations as a required element in terms of ϕ and K in the form $P[\theta_r, i(A_\lambda)]_{L_i}$, where $\theta_r \varepsilon \phi$, $A_\lambda \subset K$, and i is the isomorphism of K on the image of K, in L_i. As an example, for the ordered set L_i considered as an extension of the subset K_i, in the example 1 considered in (2.12), 1 is a required element with the representations $(a+c)_{L_i}$, $(f+c)_{L_i}$ etc., if ϕ includes finite addition.

2.32. DEF. "*The relation of a to b*, elements of an extension to ϕ of K required by ϕ and K, *is required* if either $a < b$ or $a \nless b$ is implied by their representations in terms of ϕ and k". [*M* : § 9.6].

The scope of this definition is not quite clear; the possible connotations will be discussed again in (3.51). But according to explanations given in '*M*', for the example considered above, $(a+c)_{L_i} > c$, $(f+c)_{L_i} \not> a$, are required relations.

2.33. DEF. "An extension, L, to ϕ of K is *canonical* if it consists entirely of required elements and a unique representation can be assigned to each of its formally different elements such that all the relations of L are required by their representation and K". [*M*: § 9.7].

2.4. PROPERTIES OF CANONICAL EXTENSIONS. For the canonical extension the following properties are asserted in '*M*'; their validity will be discussed in the next section. Since all except (2.45) are challenged, no proofs are given here for the propositions.

2.41. THEOREM. "*If L is a canonical extension to* ϕ *of K, then L is a minimal extension to* ϕ *of K*"; *L is also a minimal superset to* ϕ *of K.* [*M*: § 9.8, and the remarks after that].

2.42. THE FUNDAMENTAL LEMMA. "*If L is a canonical extension to* ϕ *of K, then any isomorphism which carries L into L′, a subset of itself, so that the image of K corresponds to itself, element for element, carries each element of L′ into itself, and L′ is the entire set L*". [*M*: § 9.9].

2.43. THEOREM. "*If L is a canonical extension to* ϕ *of K and* L_0 *is a minimal extension to* ϕ *of K, then L and* L_0 *are isomorphic*". [*M*: § 9.10].

2.44. THEOREM ON COMBINATION OF EXTENSIONS. "*If* L_i *is a canonical extension to* ϕ_i *of K,* L_{i_j} *a canonical extension to* ϕ_j *of* L_i*, where* ϕ_i *is a subset of* ϕ_j; *and K is a subsystem relative to* ϕ_i *of* L_i*, then* L_{i_j} *is a canonical extension to* ϕ_j *of K.*" [*M*: § 9.11].

2.45. THEOREM. "*If K is closed under* ϕ*, then K is a canonical extension to* ϕ *of itself*" [*M*: § 912].

2.46 THEOREM ON RESOLUTION OF EXTENSIONS. "*If L is a canonical extension to ϕ of K, where the only restrictions on K are that it be an ordered set closed under θ, a specified subset of ϕ, and ϕ', a subset of ϕ, contains θ, then the set L' of elements of L required by ϕ' and K is a canonical extension to ϕ' of K, and K is a subsystem relative to ϕ of L'*" [M : § 9.13].

3. CRITICAL EXAMINATION OF MACNEILLE'S THEORY OF CANONICAL EXTENSIONS.

3.1. SUMMARY OF THE MAIN CRITICISMS. My contentions regarding the results quoted in the last section from 'M', are that:

(*a*) the fundamental lemma, which is crucial for proving that a canonical extension is unique as a minimal, cannot be proved for the canonical extension defined above;

(*b*) the second part of the result (2.41), namely that a canonical extension is a minimal superset, fails to be satisfied by one of the canonical extensions constructed according to the above definition in 'M'; and

(*c*) the theorems on combination and resolution of extensions also fail in certain cases, and so are not necessary consequences of the definition of the canonical extension.

Such failures of results can be established only by considering suitably constructed examples. The above and other minor corrections are established by means of examples below.

3.2. THE CONFUSION BETWEEN 'SUBSYSTEM' AND 'SUPERSYSTEM'. We saw in (2.31) that the two relations, between an ordered set L_i and a subset K_i of it defined by the statements "L_i is a supersystem for ϕ of K_i", and "K_i is a subsystem for ϕ of L_i", are independent and distinct. But there is an apparent confusion between these two in 'M'; for, while in the constructed extentions, (see 'M' §§ 11.9, 12.7, 13.16), the extension is shown to be a supersystem for

ϕ. of the image of K, in the definition of extension it is stated that the image of K should be a subsystem of L_i for ϕ. Clearly it is the definition of extension that has to be modified, by replacing the condition 'K_i is a subsystem of L_i' by 'L_i is a supersystem of K_i'; for, with the definition given in 'M', it would be seen that an extension to ϕ of K exists only if K is itself closed for ϕ, when we would not bother to seek for extensions at all. [For if L_i is closed for ϕ, for $\theta \varepsilon \phi$ and $A \subset K_i \subset L_i$, $\theta_{L_i}(A)$ exists in L_i; so that if K_i is a subsystem for ϕ, $\theta_{K_i}(A)$ also exists in K_i.] Hence in the definition of extension, and in the later results from 'M' quoted above (in 2.21, 2.44, 2.46) such a change from 'subsystem' to 'supersystem' will be assumed to have been made in the following discussions.

3.3. THE FUNDAMENTAL LEMMA. Here we see how one of the canonical extensions constructed in 'M' fails to satisfy the fundamental lemma. The extension is of a distributive lattice to a complete Boolan algebra ['M' § 13].

EXAMPLE 2. Carrying out the extension for the distributive lattice K of the real numbers in the closed unit segment [0, 1] with the usual ordering relation between the elements, we first imbed K in a multiplicative system K'. This system K' now consists of ordered pairs (a, a') of elements of K with $a = a' = 0$ or $1 \geqslant a > a' \geqslant 0$. These pairs are ordered by a relation $<'$ defined by:

$(0, 0) <'$ every (a, a');

$(a, a') <' (b, b')$, for $(a, a') \neq (0, 0)$ if $b \geqslant a > a' \geqslant b'$, i.e. if the segment (b, b') completely encloses the segment (a, a'). Then, as shown in 'M', it would be seen that K' is a multiplicative system; that the elements $(a, 0)$ form a subset of K' isomorphic to K; that $(a, 0)$, $(1, a)$ are boolean complements* of each other in K'; and that $(a, a') = (a, 0). c(a' 0)$; c representing boolean complementation.

* In any ordered set K with units, a, a' are boolean complements of each other if $a. a' = 0$, $a \mp a' = 1$, where $\mp$ denotes the distributive sum of a and b.

Next K' is extended to $(\pi, \bar{\Sigma})$; the extension L consists of all *ideals* of K', i.e. non-null subsets of K' containing the distributive sum of every subset of itself, as well as the elements $<'$ any element in the subset. These ideals are ordered by the set inclusion relation.

It is shown in '*M*', that L is closed for $\phi = (c, \pi, \bar{\Sigma})$; that K' is a supersystem of K for ϕ, and L a supersystem of K for ϕ. Thus L is an extension to ϕ of K. The elements of L are all required, and if an ideal I of K' is assigned the representation

$$\begin{aligned} I &= \bar{\Sigma}[(a_i, a_i')], \; (a_i, a_i') \in I, \\ &= \bar{\Sigma}[(a_i . 0)] . \, c[(a_i', 0)] \\ &= \bar{\Sigma} h(a_i) . c[h(a_i')], \end{aligned}$$

h denoting the isomorphism $a \rightarrow [(a, 0)]$ between K and the image of K in L consisting of the principal ideals $[(a, 0)]$, $\{[(a, 0)]$ is the set of pairs (b, b') satisfying $(b, b') <' (a, 0)\}$. As in '*M*' it can be shown that the relations between the elements of L are the same as for the elements with similar representations in any other superset to ϕ of K (and so, in any other extension to ϕ of K).

Thus L is a canonical extension to ϕ of K. But we have an isomorphism of K' into a proper subset of itself which leaves every element of the image of K unaltered; namely, the transformation j defined by: $j(a, a') = (a, a'/2)$. It is easy to verify that j is a one-one transformation preserving order both ways between K' and its proper subset K'' consisting of elements (b, b') for which $b' = 0$ or $1 \geqslant b > 2\,b'$ [so that K'' does not contain any element of the form $(1, a)$ with $a > \frac{1}{2}$]. Aso $j(a, 0) = (a, 0)$ so that the image of K is left elementwise invariant. This isomorphism may now be extended into an isomorphism between L and a proper subset of L [not containing surely, any $[(a, a')]$ with (a, a') in $K' - K''$], by setting

$$j[I] = \{ i(a_i, a_i') \}, \; (a_i, a_i') \in I,$$

Thus there is an isomorphism of this canonical extension L into a proper subset of itself which leaves the elements in the image of L invariant. That is, L fails to satisfy the fundamental lemma.

3.4. Uniqueness of the canonical extension as a minimal extension. As MacNeille observes it is not enough if a canonical extension L to ϕ of K is just a minimal extension. For there may be another minimal L_0 such that L, L_0 are not isomorphic. That is $L \subset L_0$, $L_0 \subset L$ do not imply $L \approx L_0$. So the canonical extension must be shown to be unique as a minimal, by using other properties. But it is not observed in 'M', that conversely also $L \approx L_0$ need not imply $L \subset L_0$ and $L_0 \subset L$. So it is the simultaneous assertion $L \approx L_0$, $L \subset L_0$ and $L_0 \subset L$ which should be proved for the canonical extension L and any minimal extension L_0; or, in other words, L should be shown to be 'super-isomorphic' to L_0. The following example shows that two extensions may be isomorphic, yet neither be $\subset$ the other.

Example 3.

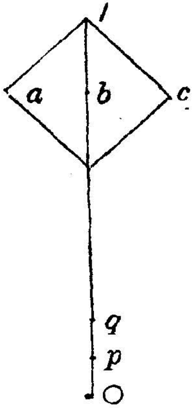

Fig. 2. K

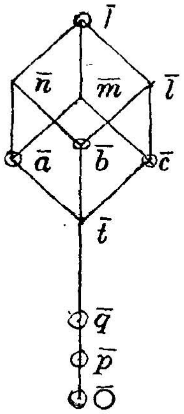

Fig. 3. L

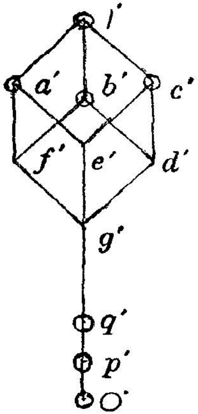

Fig. 4. L'

In the ordered set K (Fig. 2), there is no element greater than the elements of the chain p, q, r,...and less

than a, b, or c. The complete lattices $\bar{L}$, L' (Figs. 3, 4) are both extensions to $\phi = (\bar{\Sigma}, \pi)$ of K, with the encircled points in each representing the images of K. Since the products $a.b$, $a.c$, $b.c$, $a.b.c$ do not exist in K, both $\bar{L}$, L' are super-systems of the image of K for ϕ. The diagrams make it clear that $\bar{L}$, L' are isomorphic ordered sets. But it is also clear that $\bar{L}$ is not $\subset L'$, nor is $L' \subset \bar{L}$. For any isomorphism carrying $\bar{L}$ into a subset of L' and taking the image of K in $\bar{L}$ into the image of K in L', must take $\bar{a}$, $\bar{b}$, $\bar{c}$ into a', b', c', in some order; but then there is not in L' an element greater than two of a', b', c' but not the third, to correspond to $\bar{n}$ in $\bar{L}$. So such an isomorphism cannot exist. Similarly there can be no isomorphism of L' with a subset of $\bar{L}$ by which the image of K in L' is mapped on the image of K in $\bar{L}$.

3.5. THE CANONICAL EXTENSION AS A MINIMAL SUPERSET. The remark in '*M*' that a canonical extension to ϕ of K is a minimal superset to ϕ of K fails to be true for one of the extensions constructed in '*M*' itself; namely, the canonical extension of a multiplicative system to a distributive lattice described in '*M*', § 12.

EXAMPLE 4.

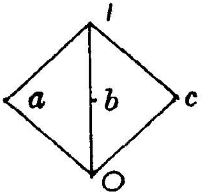

Fig. 5. K

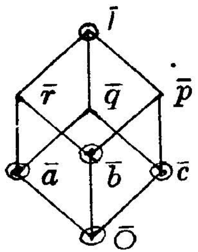

Fig. 6. $\bar{L}$

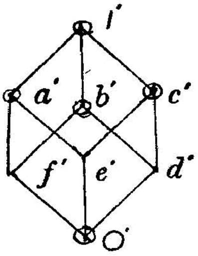

Fig. 7. L

K is the given multiplicative system, and $\bar{L}$ the canonical extension according to '*M*'. L' is also closed for

$\phi = [\pi, \Sigma]$, though it is not a supersystem of the image of K (indicated as before by encircled points) for ϕ. So L' is a superset to ϕ of K. It can be shown, as in the last example that neither $\bar{L} \subset L'$ nor $L' \subset \bar{L}$ is possible. So this canonical extension $\bar{L}$ is *not* mininal among supersets to ϕ of K, though it may be minimal among extensions to ϕ of K. The above discrepancy between the earlier statement and the constructed extension can be accounted for by observing that the scope of the definition of 'required relations' between required elements is not the same if the two results are to be true. We consider two forms of required relations:

3.51. Def. If a, b are required elements in any extension (or superset) L to ϕ of K with certain assigned representations in terms of ϕ and K *the relation of a to b is* (i) *required*; (ii) *strongly required*, if $a < b$ implies and is implied by $a' < b'$, whenever a', b' are elements with representations similar to a, b in terms of ϕ and K in any (i) extension, (ii) superset, L' to ϕ of K.

It will be seen that on similar lines, as for the proof of the minimality of the canonical extension among extensions, it could be proved that the canonical extension is a minimal superset, if the relations between the (required) elements of the canonical extension are strongly required by certain representations assigned to them. But for the extension to be a mininal extension it would be enough if the relations were required only in the weaker sense. In the example above, the relations between the elements of the canonical extension are only weakly required, and not strongly required. We may then distinguish between ordinary or weak canonical extensions and strong canonical extensions, according as the relations between the elements are weakly required or strongly required.

3.6. The combination and resolution theorems. Finally we show by examples that the combination and

resolution theorems for canonical extensions are not valid. These examples, it may be observed, really show that two of the extensions, claimed to be canonical in '*M*' on the basis of these theorems, are not even minimal extensions.

The canonical extension to (π) followed by the canonical extension to $(\pi, \bar{\Sigma})$ when applied to the ordered set K considered in example 3, leads to the extension $\bar{L}$ (Fig. 3). This, according to '*M*', should be the canonical extension of K to $(\pi, \bar{\Sigma})$. But L' is also an extension of K to $(\pi, \bar{\Sigma})$, and as we saw earlier, neither $L \subset L'$ nor $L' \subset \bar{L}$ is true. Thus $\bar{L}$ is not even a minimal extension to $(\pi, \bar{\Sigma})$ of K. This disposes of the combination theorem.

To disprove the resolution theorem we consider the following example.

EXAMPLE 5.

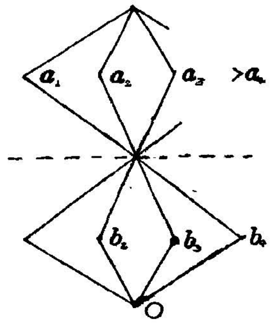

Fig. 8. K

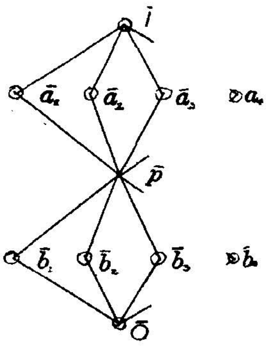

Fig. 9. $\bar{L}$

The elements $a_1, a_2, \ldots,$ in K are each $>$ every one of the elements $b_1, b_2, \ldots,$ below the dotted line; similarly the elements a'_{ij} of L' above the dotted line are all greater than each the elements b'_{kl} below the line. $\bar{L}$ is the

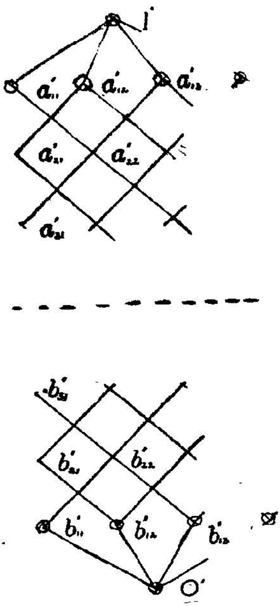

Fig. 10. L'

canonical extension of K to $\phi = [\pi, \Sigma, \times, +]$; the subset of elements of this extension required by K and the subset $\phi', = [\times, +]$, of ϕ is the entire set $\bar{L}$ itself. This, according to 'M', should be the canonical extension of K to ϕ'. But L' is an extension to ϕ' of K with the encircled points representing the elements in the image of K. It is easily seen that neither $\bar{L} \subset L'$ nor $L' \subset \bar{L}$ can be true [$L' \not\subset \bar{L}$ is obvious from the diagrams; also $\bar{L}$ cannot be $\subset L'$, as there is no element in L', corresponding to $\bar{p}$, which is $<$ every a_{ij} and $>$ every b_{ij}]. Hence $\bar{L}$ is not even a minimal extension to ϕ' of K*.

4. The modified theory of extensions.

4·1. Separation of two elements by a third. To reframe the definition of the canonical extension so that it may satisfy the fundamental lemma, the following notion of separation of elements is introduced.

4·11. Def. If a, b, c are three elements of a partially ordered set L, we say that *a and b are separated by c*, (or *c separates a, b*), if one of the following four sets of inequalities is satisfied by the elements:

(*i*) $b < c$, $a \not< c$;	(*ii*) $c \not< b$, $c < a$;
(*iii*) $b \not< c$, $a < c$;	(*iv*) $c < b$, $c \not< a$.

* In the examples 3, 4 and 5 it can be proved even that no extension of K to the operations ϕ in question can be simultaneously $\subset \bar{L}$ and $\subset L'$; that is, there can be no minimal extension to ϕ of K, and a fortiori, no canonical extension either.

It is easy to see that if $a \neq b$, then a and b are separated by each of a and b. Conversely, if a, b are separated by some element c, it can be seen that a, b should be distinct [for $a < b$ is incompatible with each of (*i*) and (*ii*), while $b < a$ is incompatible with each of (*iii*) and (*iv*)]. But for our purpose we require the following result:

4.12. LEMMA. *If an isomorphism h of L into itself maps a on b, while c separates a, b then h cannot map c on itself.*

For if c separates a, b and $h(a) = b$, where h is an isomorphism of L into itself, then one of the following four sets of inequalities is true:

(1) $b < c,\ b \nless h(c)$; (2) $c \nless b,\ h(c) < b$;
(3) $b \nless c,\ b < h(c)$; (4) $c < b,\ h(c) \nless b$.

Thus in no case can c be equal to $h(c)$.

4.2 THE EXTENSION, AND THE TWO TYPES OF CANONICAL EXTENSION. After the discussions of (2.1), and (3.5), the following are the natural forms of definitions of the extension and canonical extension. The one extra condition in the definition of canonical extension is introduced to make the canonical extension satisfy the fundamental lemma.

4.21. DEF. A superset, L, to ϕ of K is an extension to ϕ of K, if it is a supersystem for ϕ of the image of K.

4.22. DEF. An extension L to ϕ of K in which every element is required by ϕ and K is (1) a (weak) canonical extension, (2) a strong canonical extension to ϕ of K, if the elements can be assigned representations such that the relation between every two of them is required in all (1) extensions to ϕ of K, (2) supersets to ϕ of K; and, finally, if every two distinct elements of L are separated by some element in the image of K.

4.23. As we remarked in (3.5), we now show that the weak and strong canonical extensions, even without the last condition on the separation of elements of L by elements in the image of K, are respectively a minimal extension and a minimal superset. In fact, we have even

a stronger result; namely, that *if L' is any other extension, or superset to the same set, ϕ of operations, with K' as image of K, and h is any isomorphism between the image of K in L and K', then h can be extended into an isomorphism between L and a subset of L' by associating to each element of L the element of L' with a similar representation*; i.e. to an element, $P[\theta_r, g(A_\lambda)]_L$,* of L, where g is the isomorphism from K to the image of K in L, we associate the element $P[\theta_r, g'(A_\lambda)]_{L'}$ of L', where g' is the isomorphism from K to the image of K in L'. For the isomorphism h from $g(K)$ to $g'(K)$, we may take $(g'.g^{-1})$ itself. [That this extension of h is an isomorphism, in both the cases, follows from the fact that the relations between every two of the elements of L are required, or strongly required, by their representations.]

4·3. The fundamental properties of the canonical extension.

4·31. After the considerations of § 3·4, we now seek to prove that the canonical extension is minimal and unique as such to within superisomorphisms; in other words, we have the

Theorem. (*a*) *A canonical extension to ϕ of K is a minimal extension to ϕ of K, and is super isomorphic with every other minimal extension*;

(*b*) *a strong canonical extension to ϕ of K is a minimal superset to ϕ of K, and is super isomorphic with every other minimal superset.*

Proof. We already saw in the last section that the canonical extension, L, is a minimal extension. If L_0 is any other minimal extenison, there is an isomorphism g of L_0 into a subset of L under which the image of K in L_0 is mapped on the image of K in L. As in (4·23), the isomorphim $h = g^{-1}$ between the images of K in L and L_0 may be extended into an isomorphism h between L and a subset of L_0. Then $g.h$ is an isomorphism of L

* $P[\theta_r, g(A_\lambda)]$ denotes a succession of the finite set of operations (θ_r) from ϕ applied in a specified manner and order upon the subsets $g(A_\lambda)$ of $g(K)$, where the (A_λ) are subsets of K.

into a subset of itself which leaves the elements of the image of K invariant. Hence $(g.h)$ must take each element of L into itself. But this means that h, g are both one-one transformations and inverses of each other. Thus h is a super isomorphism between L and L_0. This proves part (a).

A similar proof is possible for part (b).

4.32. The theorem on combination of extensions can be stated in the following weaker form :

THEOREM. *If L_i is a canonical extension to ϕ of K where L_i is also a supersystem of the image, $i(K)$ of K for $\phi', \supset \phi$, and L_j is a strong canonical extension to ϕ' of L_i, with $h(L_i)$ as image of L_i, then L_j is a minimal extension to ϕ' of K with $h\,i\,(K)$ as image of K. If it is not a canonical extension to ϕ' of K, no canonical extension to ϕ' of K can exist.*

PROOF. It is easy to see that L_j is an extension to ϕ' of K with $hi\,(K)$ as image K. Also all the elements of L_j are required by ϕ and K.

If $\bar{L}$ is now any extension to ϕ' of K, with $\bar{K} = t(K)$ as image of K, it is also an extension to ϕ $(\subset \phi')$ of K. So, as in (4.23), the isomorphism $t.i^{-1}$ from $i(K)$ to $t(K)$ can be extended into an isomorphism τ from L_i to a subset of L_j. Hence L_j is a superset to ϕ' of L_i with $\tau(L_i)$ as image of L_i. As in (4.23) again, the ismorphism $\tau.h^{-1}$ from $h(L_i)$ to $\tau(L_i)$ can be extended into an isomorphism τ' between the strong canonical extension L_j, of L_i to ϕ, and a subset of $\bar{L}$. Thus $L_j \subset \bar{L}$, or L_j is a minimal extension to ϕ' of K.

By the last theorem (4.31), if a canonical extension to ϕ' of K should exist it should be super-isomorphic to this minimal L_j, and so L_j itself should be a canonical extension to ϕ' of K. So if it is not, no canonical extension to ϕ' of K can exist.

We may observe here that this theorem is inapplicable to the case considered in (3.6) as the second canonical

extension there is not a strong one. Thus we find there that the combination does not lead to even a minimal extension.

4·33. We can similarly give the following modified form of the resolution therem.

THEOREM. *If the elements of a canonical extension, L, to* ϕ *of K which are required by* ϕ', $(\subset \phi)$, *and K can be assigned representations in terms of* ϕ' *and K so that the relation between every two of them is required by their representations (in all extensions to* ϕ' *of K), and if the set K′ of these required elements is a subsystem of L for* ϕ', *then L′ is a canonical extension to* ϕ' *of K.*

The proof is quite simple; we may observe that the separation condition is automatically satisfied for the elements of L', as it is true for the elements of L.

4·34. We give below two more results regarding canonical extensions which will be of use in the second part of this paper. Their proofs are elementary.

4·341. THEOREM OF DUALITY. *If* $L, \bar{L}$ *are dual ordered sets,* $K, \bar{K}$ *are also dual ordered sets, and* ϕ, $\bar{\phi}$ *are sets of operations each consisting of all operations dual to those in the other, then whenever L is any form of superset or extension to* ϕ *of K so is* $\bar{L}$ *to* $\bar{\phi}$ *of* $\bar{K}$, *and vice-versa.*

4·342. THEOREM: *If a canonical extension* L_i *to* ϕ *of K is also an extension to a second set of operations* θ, *of K with the same image of K, then* L_i *is a canonical extension to* $(\phi \cup \theta)$ *of K, with the same image of K. If in the above,* L_i *is a strong canonical extension to* ϕ *of K, it is also a strong canonical extension to* $(\phi \cup \theta)$ *of K.*

A POWER SERIES THAT CONVERGES AND DIVERGES AT EVERYWHERE DENSE SETS OF POINTS ON ITS CIRCLE OF CONVERGENCE.

BY

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It is known that the behaviour of a power series on the circumference of its circle of convergence can be very complicated*, Friedrich Lösch showed that the series †

$$\sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{1}{2}\, x\,(1+x^{2^n})\right]^{2^{2^n}}$$

converges at an everywhere dense set of points on the unit circle and diverges at another everywhere dense set of points. The object of this note is to give a remarkably simple example of a power series having that property.

Let $\lambda_n = 3^n$ or 2.3^n according as n is odd or even; then we have the

* It may converge at all points of the circumference and uniformly too and yet not absolutely; we may have that $a_n \to 0$ and yet the series may not converge at one point even on the unit circle; or it may converge at precisely one point on the circumference. Again, the points of divergence of the series may precisely be an arbitrarily given enumerable set of points on the circle of convergence.

Mention may also be made of a theorem of P. Fatou and M. Riesz: *If $a_n \to 0$ then the series $\Sigma a_n x^n$ is convergent at every point on the unit circle at which the function $f(x) = \Sigma a_n x^n$ is regular; moreover the convergence is uniform over any arc if the function is regular at all points of the arc including the end points.*

† *Mathematische Zeitschrift*, 34 (1931), 291-26.

THEOREM. *The series*

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^{\lambda_n}}{n}=-\frac{x^3}{1}+\frac{x^{18}}{2}-\frac{x^{27}}{3}+\frac{x^{162}}{4}-\frac{x^{243}}{5}+\dots$$

converges at an everywhere dense set of points on the unit circle and diverges at another everywhere dense set of points on that circle.

PROOF. Suppose that x is a root of the equation $x^{3^m}=1$ for an integral value of m; then plainly $x^{\lambda_n}=1$ whenever $n \geqslant m$ and it follows that the series is convergent. Also, evidently, the roots of the equations $x^{3^m}=1$ $(m=1, 2, 3, \dots)$ are everywhere dense on the unit circle. Nextly, if $x^{3^m}=-1$ then we see that

$$(-1)^n\frac{x^{\lambda_n}}{n}=\frac{1}{n}$$

whenever $n \geqslant m$ and the series is divergent, and the set of points x for which $x^{3^m}=-1$ for some positive integral value or other is everywhere dense on the unit circle. This completes the proof of the theorem.

This example will also come in handy in the class room when one is lecturing on the theorem of Abel that *if* Σa_n *is convergent then* $\lim\limits_{x\to 1-0} \Sigma a_n x^n = \Sigma a_n$. In this theorem we see that x is made to approach 1 along the real axis from the left side. While we can refer to known generalizations in which the path of approach of x to the point 1 is less resticted, we have also to point out that some sort of restriction is necessary. For this latter purpose the example of this note is useful since it will be seen that if m is an integer, and ξ be such that $\xi^{3^m}=-1$ then we have

$$f(x)=\Sigma(-1)^n\frac{x^{\lambda_n}}{n}\to\infty$$

as $x\to\xi$ radially since the terms are real and positve for $n \geqslant m$. Since such points ξ are everywhere dense on the unit circle we can choose points $\xi_1, \xi_2, \xi_3, \dots$ such that

$|\xi_n| < 1$, $n = 1, 2, 3, \ldots$, $\xi_n \to 1$ as $n \to \infty$, $|f(\xi_n)| > n$, $n = 1, 2, 3, \ldots$. If now $x \to 1$ along the path obtained by drawing straight lines to join ξ_n, ξ_{n+1} $(n = 1, 2, 3, \ldots)$ then plainly we get a path lying inside the unit circle but which is such that as $x \to 1$ along the path, the function $f(x)$ is unbounded and therefore does not tend to $\Sigma a_n = -1 + \frac{1}{2} - \frac{1}{3} \ldots = -\log 2$.

I give below another example of a power series which enables one to sharpen the theorem slightly. Let $\mu_n = 1.3 \ldots (2n-1)$ for $n = 1, 2, 3, \ldots$. We take $\nu_n = \mu_n$ or $2\mu_n$ according as n is odd or even. We shall now prove that

$$\sum \frac{1}{n} x^{\nu_n}$$

converges on any arc of the unit circle at a set of points which has the power of the continuum and diverges at a similar set.

Let $\theta = \sum_{n=1}^{\infty} a_n/\mu_n$, where $a_1, a_2, a_3, \ldots$ are integers, $a_1 = 0$ or 1, $0 \leqslant a_n < \mu_n/\mu_{n-1} = 2n-1$, $n = 2, 3, 4, \ldots$, $\lim_{n\to\infty} a_n < \infty$, and a_n is odd for a finite odd number of values of n and even for all other values of n. It is easy to verify that the set of such numbers θ in any interval of $(0, 2)$ has the power of the continuum. If $\mu_n \theta = i_n + f_n$, where i_n is an integer and $0 \leqslant f_n < 1$ then i_n is an odd integer for all $n \geqslant N = N(\theta)$ and $f_n = O(1/n)$; for, whenever n is large we have

$$i_n = a_1 \frac{\mu_n}{\mu_1} + a_2 \frac{\mu_n}{\mu_2} + \ldots + a_n \frac{\mu_n}{\mu_n} \equiv a_1 + a_2 + \ldots + a_n \pmod 2,$$

$$0 \leqslant f_n = \sum_{m=n+1}^{\infty} \frac{a_m \mu_n}{\mu_m} \leqslant M\left(\frac{1}{2n+1} + \frac{1}{(2n+1)(2n+3)} + \ldots\right) < \frac{M}{n},$$

where M is the upper bound of the numbers a_m, $m = n+1, n+2, \ldots$

Hence if $x = e^{\pi i\theta}$ then for all large values of n we have

$$x^{\mu_n} = e^{(i_n + f_n)\pi i} = -1 + O\left(\frac{1}{n}\right),$$

and it follows that

$$x^{\nu_n} = (-1)^n + O\left(\frac{1}{n}\right), \frac{1}{n} x^{\nu_n} = \frac{(-1)^n}{n} + O\left(\frac{1}{n^2}\right),$$

and hence the series $\sum \frac{1}{n} x^{\nu_n}$ is convergent. Similarly if $n = e^{2\pi i\theta}$ then it follows that $x^{\nu_n} = 1 + O\left(\frac{1}{n}\right)$ and that $\sum \frac{1}{n} x^{\nu_n} = \infty$. From what has been said about the numbers θ it follows that the set of numbers x on any arc of the unit circle has in each of the two cases (of convergence and divergence) the power of the continuum.

I may add, finally, that for the series under consideration the set of points of non-convergence on the unit circle is of Lebesgue measure zero, for since the sum of the squares of the moduli of the coefficients is convergent it follows that the real and imaginary parts are Fourier series, and therefore summable $(C, 1)$ almost everywhere. Being a lacunary series we have that the above summability implies convergence.

A CHARACTERIZATION OF MAXIMAL-HAUSDORFF SPACES.*

BY

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1. INTRODUCTION. In my paper on Maximal-Hausdorff spaces† I have proved the existence of non-bicompact maximal-Hausdorff spaces. The purpose of the present paper is to give a final and more satisfactory characterization of maximal-Hausdorff spaces by proving that *the Maximal-Hausdorff spaces are identical with the H-closed semi-regular spaces.* A part of this result has already been proved [see Paper I Theorems III and IV]. The known result that *the Hausdorff bicompact spaces are maximal-Hausdorff* is a special case of this result, since *the Hausdorff bicompact spaces are H-closed semi-regular spaces.*

2. DEFINITIONS. We shall denote by $(R;f)$ the topological space R whose topological structure is f, where the neighbourhoods of any point $p \in R$ are the open sets of $(R;f)$ containing the point. We shall denote by $f(p)$ any neighbourhood of the point $p \in R$ in the f-topology.

A Hausdorff space R will be said to be *locally maximal-Hausdorff* at a point $p \in R$, if the space cannot be strengthened at p without rendering p H-inseparable from some other point of R.

3. THEOREM I. *A Hausdorff space which is locally maximal-Hausdorff at some point of it is neccssarily H-closed.*

This result is clearly stonger than that of Theorem III of Paper I, while the proof is practically on the same

* My gratitude is due to Dr. R. Vaidyanathaswamy, for his very valuable suggestions and criticisms in the preparation of this paper. This was read at the 15th conference of the Ind. Math. Soc. at Waltair (Dec. 1947).

† *The Proceedings of the Indian Academy of Sciences*, 26 (1947), Sec. A. This paper will be referred to, in what follows, as Paper I.

lines. It will be noticed that the present theorem is true quite irrespective of the fact whether the point p is semi-regular or irregular without being semi-regular.

PROOF. Let $(R;f)$ be a Hausdorff space which is locally maximal-Hausdorff at a point $p \in R$. Let us assume that $(R;f)$ is not H-closed. Then there exists an open covering $\{G\}$ of R such that, for every choice of the finitely-many elements $G_1, G_2, \ldots, G_n$ of $\{G\}$, the open set $F = (R - \sum_1^n \bar{G}_i)$ is non-null. Now the family of open sets $\{F\}$ are such that the intersection of any two F's is an F. Moreover, if we choose for a neighbourhood $f_0(x)$ of any point $x \in R$, an element of $\{G\}$ which contains x then $f_0(x)$ is disjoint with $[R - \overline{f_0(x)}]$ which, by definition, is an F. Hence there is at least one F which is disjoint with an $f(x)$ whatever $x \in R$ may be. Let us now define a topological space $R + \xi$ by the addition of an abstract point ξ, by taking the neighbourhoods of points of R to be the open sets of $(R;f)$ containing these points and prescribing the neighbourhoods of ξ to be the sets $\{\xi + F\}$. The space $R + \xi$ is easily seen to satisfy the postulates H_1, H_2, H_3. The space $R + \xi$ is a topological space in which the given space $(R;f)$ is topologically imbedded. Since there is an $f(\xi)$ disjoint with a neighbourhood of every point of R, we see that $R + \xi$ is a Hausdorff space. Further, since the F's are non-null, the point ξ is contact point of R in $R + \xi$. Now, let us topologically identify ξ with p and let $(R;\phi)$ be the space thus obtained. Since p and ξ are H-separable in $R + \xi$, we see that ϕ is stronger than f; and in $(R;\phi)$, p is H-separable from every other point of R. Thus ϕ is a Hausdorff topology stronger than f at p and equivalent to f at all other points of R. This contradicts the hypothesis that $(R;f)$ is locally maximal-Hausdorff at the point p. It follows, therefore, that our assumption is false and so $(R;f)$ must be H-closed.

As a special case of this theorem we see that *A Hausdorff space which is locally maximal-Hausdorff at a semi-regular point of it is necessarily H-closed.*

Conversely, we have

THEOREM II. *An H-closed space is locally maximal-Hausdorff at every semi-regular point of it.*

PROOF. Let $(R; f)$ be an H-closed space and p a semi-regular point of it. Then the given system $\{f(p)\}$ of neighbourhoods of p is equivalent to the system of open domain-neighbourhoods of p. For, every open domain-neighbourhood of p, being an absolute neighbourhood of p in the given topology, is an $f(p)$. And every $f(p)$ contains, since p is semi-regular, the interior of the closure of another $f(p)$. But, the interior of the closure of any $f(p)$ is an open domain-neighbourhood of p.* Hence, we may, without loss of generality, assume all $f(p)$'s to be open domain-neighbourhoods of p. Let ϕ be a T_1-topology, stronger than f, on R, obtained by strengthening the given neighbourhood system at the point p only, the given neighbourhood system being unaltered at the remaining points of R. Then, there is at least one $f(p)$, say $f_0(p)$ which contains no $\phi(p)$ and has, therefore, p as boundary point in ϕ-topology. Thus, p is contact point of $[R-f_0(p)]$ in ϕ-topology. Since $f_0(p)$ is an open domain, it has the same frontier as Ext. $f_0(p)$. Since ϕ is stronger than f, every $\phi(p)$ is an open set in the f-topology. Now, *every* $\phi(p)$ *must have a non-null intersection with Ext.* $f_0(p)$. For, if a $\phi(p)$. Ext. $f_0(p)=0$, then that $\phi(p)\subset\overline{f_0(p)}$ and therefore $\subset$ Int. $\overline{f_0(p)}=f_0(p)$, contradicting the fact that $f_0(p)$ contains no $\phi(p)$. Hence, if we denote the intersections $\{\phi(p)$. Ext. $f_0(p)\}$ by $\{F\}$, these form a family of non-null open sets of $(R; f)$. Further, since the intersection of any two $\phi(p)$'s is a $\phi(p)$, it follows that the intersection of any two F's is an F. Now add an abstract

* *Vide* R. Vaidyanathaswamy, *Treatise on set Topology*, Part I, (1947), page 91.

point ξ to R and define $R+\xi$ as a topological space by taking the neighbourhoods of points of R to be open domains of R containing these points and prescribing the neighbourhoods of ξ to be the sets $\{\xi+F\}$. The space $R+\xi$ thus defined is easily seen to satisfy the postulates H_1, H_2, H_3. The space $R+\xi$ is a topological space in which the given space $(R\,;\,f)$ is topologically imbedded.

Since the topology ϕ is T_1, there is a $\phi(p)$ and therefore an F, disjoint with any point q of $R-p$. Since $f_0(p)$ contains no $\phi(p)$, it follows that there is an F disjoint with $f_0(p)$. Hence, in the space $R+\xi$, for any point $x \varepsilon R$, there is an $f(\xi)$ disjoint with x, and further, there is an $f(\xi)$ disjoint with $f_0(p)$. Thus, $R+\xi$ is a T_1-space in which ξ and p are H-separable. Since the F's are non-null, ξ is contact point of R in $R+\xi$, so that R is non-closed in $R+\xi$. Now the relative topology of R in $R+\xi$ is the same as the given topology f of R; and $(R\,;\,f)$ is given to be H-closed. Hence, $R+\xi$ is not a Hausdorff space, so that ξ is not H-separable with at least one point q $(\neq p)$ of R. Thus, there is at least one point q $(\neq p)$ of R, every neighbourhood of which intersects every F and therefore intersects every $\phi(p)$. Hence the space $(R\,;\,\phi)$ is not Hausdorff, i.e. $(R\,;f)$ is locally maximal-Hausdorff at the point p.

NOTE 1. The set of points $\{q\}$ from which ξ is not H-separable in Theorem II, always contains those irregular semi-regular points of R from which p is U-inseparable in the given topology f of R (vide Paper I. Theorem IV).

NOTE 2. Theorem II is true whether the point p is an irregular semi-regular point or a regular point.

We now deduce immediately

THEOREM III. *An H-closed semi-regular space is maximal-Hausdorff.*

For, an H-closed semi-regular space is locally maximal-Hausdorff at every point of it.

4. REMARKS. From the results proved in Paper I it is clear that an irregular maximal-Hausdorff space should satisfy the following conditions:—

(i) *it must be H-closed*;

(ii) *it must be semi-regular* (*without being regular*);

(iii) *every irregular semi-regular point must be U-inseparable from at least one other irregular semi-regular point.*

We have now proved that the necessary and sufficient conditions for any Hausdorff space to be maximal Hausdorff is that it should be *H*-closed and semi-regular. We conclude, therefore, that (iii) is a necessary consequence of (i) and (ii).

Now, a Hausdorff space with a single irregular semi-regular point cannot be maximal Hausdorff (*vide* Paper I Th. IV, Cor. 1). Hence, such a space cannot be *H*-closed, although a Hausdorff space with a single irregular (but not semi-regular) point can be *H*-closed (*vide* Paper I Examples 1 and 2).

We shall now prove directly

THEOREM IV. *A Hausdorff space with a single irregular semi-regular point* (*the remaining points being regular*) *cannot be H-closed.*

PROOF. Let $(R; f)$ be a Hausdorff space and p the only irregular semi-regular point of it, the remaining points being regular. We may take as before that all $f(p)$'s are open domain-neighbourhoods of p. Let $\phi(p)$ represent the system of regular neighbourhoods of p and let ϕ represent the topology obtained by replacing $\{f(p)\}$ by $\{\phi(p)\}$, the neighbourhood systems f and ϕ being the same at all other points of R. We have seen* that $(R; \phi)$ is a Hausdorff space and that ϕ is stronger than f. Hence there is at least one $f(p)$, say $f_0(p)$, which contains no $\phi(p)$ and has therefore p as boundary point in ϕ-topology. Thus, p is contact point of $[R-f_0(p)]$ in ϕ-topology. Since $f_0(p)$ is an open domain, it has the

* *Vide* Paper I, Theorem IV.

same frontier as Ext. $f_0(p)$. Since ϕ is stronger than f, every $\phi(p)$ is an open set in the f-topology. As in Theorem II, we see that every $\phi(p)$ has a non-null intersection with Ext. $f_0(p)$. Denoting the intersections $\{\phi(p). \text{Ext.} f_0(p)\}$ by $\{F\}$, we see, as before, that the intersection of any two F's is an F. We now define the space $R+\xi$ by taking the neighbourhoods of points of R to be open domains of $(R;f)$ containing these points and prescribing the neighbourhoods of ξ to be the sets $\{\xi+F\}$. The space $R+\xi$ is a topological space satisfying H_1, H_2, H_3 and in which the given space $(R;f)$ is topologically imbedded.

Since the topology ϕ is Hausdorff, there is a $\phi(p)$, and therefore an F, disjoint with a neighbourhood of any point $q \epsilon R-p$. Since $f_0(p)$ contains no $\phi(p)$, there is an F disjoint with $f_0(p)$. Hence, in the space $R+\xi$, for any point $x \epsilon R$, there is an $f(x)$ and an $f(\xi)$ such that $f(x).f(\xi)=0$. Thus $R+\xi$ is a Hausdorff space. Since the F's are non-null, ξ is contact point of R in $R+\xi$, so that R is non-closed in $R+\xi$. Now, the relative topology of R in $R+\xi$ is the same as the given topology f of R and $(R;f)$ is homeomorphic to a non-closed subset of the Hausdorff space $R+\xi$. Hence $(R;f)$ cannot be H-closed.

We may similarly prove

THEOREM V. *If a Hausdorff space contains at least one irregular semi-regular point, which is U-separable from every other irregular point of the space, then the space cannot be H-closed.*

For, let $(R;f)$ be a Hausdorff space in which the point p is an irregular semi-regular point which is U-separable from every other irregular point of the space. We may take, as before, that every $f(p)$ is an open domain-neighbourhood of p. Let $\{\phi(p)\}$ represent the system of regular neighbourhoods of p and let ϕ represent the

topology obtained by replacing $\{f(p)\}$ by $\{\phi(p)\}$, the neighbourhood systems f and ϕ being the same at all other points of R. As before, the space $(R;\phi)$ is a Hausdorff space and ϕ is stronger than f. From this stage the proof is quite similar to that of Theorem IV above.

Lastly, we may notice that the class of H-closed Urysohn spaces includes that of regular maximal-Hausdorff spaces, i.e. Hausdorff bicompact spaces; and the class of H-closed non-Urysohn spaces includes that of irregular maximal-Hausdorff spaces, i.e. H-closed semi-regular non-regular spaces, where a non-Urysohn space is one which contains at least one pair of U-inseparable points.

We may now easily infer that *The class of H-closed Urysohn semi-regular spaces is identical with the class of regular maximal-Hausdorff spaces, i.e. identical with the class of Hausdorff bicompact spaces.*

DEFINITION. A point of a topological space will be called a *thoroughly irregular point* of the space if it is irregular without being semi-regular.

It is clear that every point of a H-closed Urysohn space is either regular or thoroughly irregular.

We now prove a theorem regarding H-closed Urysohn irregular spaces, viz.

THEOREM VI. *If $(R;f)$ is an H-closed Urysohn space and p is a thoroughly irregular point of it, then every Int. $f(p)$ is a regular neighbourhood of the point p.*

PROOF. Let $\{\phi(p)\}$ represent the system of regular neighbourhoods of p and let ϕ represent the topology obtained by replacing $\{f(p)\}$ by $\{\phi(p)\}$, the neighbourhood systems f and ϕ being the same at all other points of R. Then, as above, we see that $(R;\phi)$ is a Hausdorff space and that ϕ is stronger than f. Now, no irregular neighbourhood of p can contain a $\phi(p)$. Hence every

irregular neighbourhood of p has p as boundary point in ϕ-topology. Let $f_0(p)$ be one such neighbourhood of p. Since ϕ is stronger than f, every $\phi(p)$ is an open set in the f-topology. Now, if every $\phi(p)$ has a non-null intersection with Ext. $f_0(p)$, we can construct the Hausdorff space $R+\xi$, as before, in which R would be non-closed and therefore R would not be H-closed. Hence, there is at least one $\phi(p)$ disjoint with Ex. $f_0(p)$, so that this $\phi(p) \subset \overline{f_0(p)}$ and therefore $\subset$ Int. $\overline{f_0(p)}$, which is impossible unless Int. $\overline{f_0(p)}$ is a regular neighbourhood of p in f-topology.

Now, Int. $\overline{\phi(p)}$ always contains $\phi(p)$ and, hence, is always a regular neighbourhood of p in f-topology. Hence the theorem.

An example of an H-closed Urysohn irregular space is afforded by Ex. 1 of Paper I where the point 0 is thoroughly irregular and the remaining points are regular.

Now if a pair of points p, q are U-inseparable in a Hausdorff space $(R; f)$ then both the points are irregular.

For, suppose p is regular. Since the given space is Hausdorff, there exist a pair of neighbourhoods $f(p)$, $f(q)$ of p and q such that $f(p).f(q) = 0 = f(p).\overline{f(q)}$. If p were regular, then there would be a neighbourhood $f_1(p)$ of p such that $\overline{f_1(p)} \subset f(p)$; and $\overline{f_1(p)}.\overline{f(q)} = 0$ contradicting the hypothesis that p and q are U-inseparable.

Thus we see that *the irregular maximal-Hausdorff spaces are identical with the H-closed non-Urysohn semi-regular spaces.*

FUNCTIONS OF INTEGRABLE FUNCTIONS

BY

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In certain analytical considerations (notably, in statistics, in connection with the definition of unbiased estimates) one encounters expressions such as $\int_a^b f(x(t))\,dt$. It is, of course, necessary in these cases to establish the integrability of the composite integrand. The purpose of this note is to present two simple theorems concerning such integrability problems. Since it is known that a Borel measurable function (i.e. a Baire function) of a measurable function is measurable, whereas an arbitrary (Lebesgue) measurable function of a measurable function need not be measurable, it is desirable, in order to avoid pathological questions, to restrict attention to the case in which f is Borel measurable.

DEFINITION. A real valued function f of n real variables is a $(p \to q)$ function on an interval (a, b) if and only if for every set $(x_1, \ldots, x_n)$ of n real valued functions defined on (a, b) and belonging to $L_p(a, b)$ the function g, defined by $g(t) = f(x_1(t), \ldots, x_n(t))$, belongs to $L_q(a, b)$.

THEOREM 1. *A necessary and sufficient condition that a Borel measurable function f be a $(p \to q)$ function on $(0, 1)$ is that there exist a positive constant K such that for all points $(y_1, \ldots, y_n)$ the inequality*

$$|f(y_1, \ldots, y_n)|^q \leqslant K(1 + |y_1| + \ldots + |y_n|)^p \qquad (1)$$

is satisfied.

Proof. The sufficiency of the condition is obvious. To prove necessity, suppose that f is a Borel measurable $(p \to q)$ function and that (1) is violated. It follows that there exists a sequence $\{(y_1^k, \dots, y_n^k)\}$ of points such that for $k = 1, 2, \dots,$

$$|f(y_1^k, \dots, y_n^k)|^q > k(1 + |y_1^k| + \dots + |y_n^k|)^p.$$

Since the infinite series

$$s = \sum_{k=1}^{\infty} k^{-2}(1 + |y_1^k| + \dots + |y_n^k|)^{-p}$$

is convergent, it follows that

$$0 < p_k = \frac{1}{s} k^{-2}(1 + |y_1^k| + \dots + |y_n^k|)^{-p} < \infty;$$

clearly

$$\sum_{k=1}^{\infty} p_k = 1.$$

It is therefore possible to find a disjoint sequence $\{E_k\}$ of subintervals of the unit interval such that $m(E_k) = p_k$. If, for $i = 1, \dots, n$, the functions x_i are defined by

$$x_i(t) = \begin{cases} y_i^k & \text{for } t \in E_k, \\ 0 & \text{for } t \,\varepsilon'\, E_k, \end{cases}$$

then

$$\int_0^1 |x_i(t)|^p \, dt = \sum_{k=1}^{\infty} p_k |y_i^k|^p =$$

$$= \frac{1}{s} \sum_{k=1}^{\infty} k^{-2} \left(\frac{|y_i^k|}{1 + |y_1^k| + \dots + |y_n^k|} \right)^p.$$

Since the last written series is convergent, $x_i \in L_p(0, 1)$ for $i = 1, \dots, n$. On the other hand

$$\int_0^1 |f(x_1(t), \ldots, x_n(t))|^q \, dt = \sum_{k=1}^{\infty} p_k |f(y_1^k, \ldots, y_n^k)|^q >$$

$$> \frac{1}{s} \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

so that if $g(t) = f(x_1(t), \ldots, x_n(t))$, then $g \,\varepsilon'\, L_q$ (0, 1). This contradiction completes the proof of Theorem 1.

THEOREM 2. *A necessary and sufficient condition that a Borel measurable function f be a $(p \to q)$ function on $(-\infty, +\infty)$ is that there exist a positive constant K such that for all points $(y_1, \ldots, y_n)$ the inequality*

$$|f(y_1, \ldots, y_n)|^q \leqslant K(|y_1| + \ldots + |y_n|)^p \tag{2}$$

is satisfied.

PROOF. The sufficiency of the condition is again obvious. To prove necessity, suppose that f is a Borel measurable $(p \to q)$ function and that (2) is violated. It follows that there exists a sequence $\{(y_1^k, \ldots, y_n^k)\}$ of points such that, for $k = 1, 2, \ldots,$

$$|f(y_1^k, \ldots, y_n^k)|^q > k(|y_1^k| + \ldots + |y_n^k|)^p.$$

Since the function which is identically zero belongs to L_p, the fact that f is a $(p \to q)$ function implies that the constant function $f(0, \ldots, 0)$ belongs to L_q. It follows that $f(0, \ldots, 0) = 0$ and, consequently, that none of the points $(y_1^k, \ldots, y_n^k)$ is the origin. Since, therefore,

$$0 < p_k = k^{-2} (|y_1^k| + \ldots + |y_n^k|)^{-p} < \infty,$$

it is possible to find a disjoint sequence $\{E_k\}$ of intervals such that $m(E_k) = p_k$. If, for $i = 1, \ldots, n$, the functions x_i are defined by

$$x_i(t) = \begin{cases} y_i^k \text{ for } t \,\varepsilon\, E_k, \\ 0 \text{ for } t \,\varepsilon'\, E_k, \end{cases}$$

then

$$\int_{-\infty}^{+\infty} |x_i(t)|^p dt = \sum_{k=1}^{\infty} p_k |y_i^k|^p < \infty,$$

and therefore $x_i \varepsilon L_p(-\infty, +\infty)$ for $i = 1, \ldots, n$. On the other hand

$$\int_{-\infty}^{+\infty} |f(x_1(t), \ldots, x_n(t))|^q dt = \sum_{k=1}^{\infty} p_k |f(y_1^k, \ldots, y_n^k)|^q >$$

$$> \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

so that if $g(t) = f(x_1(t), \ldots, x_n(t))$, then $g \varepsilon' L_q(-\infty, +\infty)$. This contradiction completes the proof of Theorem 2.

The considerations of Theorems 1 and 2 remain valid for non atomic measure spaces instead of intervals; the case of atomic measure spaces reduces to delicate (and as yet unanswered) convergence questions for certain infinite series.

ON SKEWFIELDS OF A GIVEN DEGREE

BY

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In a recent paper (*), the author has introduced the notion of *rank of commutativity* of a ring. This notion will be applied to skewfields in the present note. It will be shown that the degree of a skewfield S can be characterized without explicit reference to the centre of S or to a basis of S over its centre. In this way one obtains formulas interconnecting the "determinants" used for characterizing the rank of commutativity of S with the commutator group of S', where S' denotes the multiplicative group formed by the non-zero elements of S. For the simplest non-trivial case, *degree* $S = 2$, we obtain particular results.

1. The notations used in the paper quoted above and the results, as far as they will be used in the present note, are being repeated at first for the reader's convenience. By

$$[a_1, \ldots, a_r] = \Sigma \pm a_{i_1} \cdots a_{i_r} \tag{1}$$

we denote the sum of the $r!$ products which we obtain by permuting the indices using the positive sign for the even permutations and the negative sign for the odd permutations. The formula (1) can therefore be considered as a determinant with r equal rows, the elements belonging to a ring which in general is non-commutative. The laws of commutative determinant theory about permutation of columns, column addition and multiplication of a column with an element of the centre hold also in the present case. If $a_1, \ldots, a_r$ are linearly dependent over the centre, then the determinant (1) is equal to zero, but the converse statement does not hold. Let the a's run over

* Über den Commutativitätsrang in einem Ringe, to be published in *Mathematische Annalen*.

the elements of a ring R; when for $r > m$, all the determinants (1) are equal to zero, but there exists a determinant with m columns which is different from zero, then m is called the *rank of commutativity* of R.

$$m = roc\ (R). \qquad (2)$$

Let R be a simple ring with unit element (the semi simple case can be derived therefrom easily) and Z its centre, then $[R:Z] = n^2$ and it can be proved* that

(1) Every determinant (1) in which an element a belongs to the centre, is equal to zero when r is even;

(2) $roc\ (R)$ is an odd number;

(3) $roc\ (R)$ depends on n and the characteristic p (prime number or zero) only and

$$roc\ (R) = roc\ (n, p); \qquad (3)$$

(4) $2n-1 \leqslant roc\ (n, p) = roc\ (n, 0) < n^2$; (4)

(5) $2 + roc\ (n, p) \leqslant roc\ (n+1, p)$; (5)

(6) $roc\ (2, p) = 3$; $roc\ (3, p) = 5$. (6)

It is not yet known whether for some $n > 3$, the rank of commutativity is $> 2n-1$ and whether it actually depends on p. However it follows already from (5) that for a given characteristic, the $rank[R:Z] = n^2$ is uniquely determined by $roc(R)$. In particular

$$roc\ (R) = 3 \text{ implies } n = 2,$$
$$roc\ (R) = 5 \text{ implies } n = 3.$$

2. As a skewfield S is a simple ring with unit element, the above results can be applied to $R = S$. In this case, the number n is called the *degree* of S. Every determinant (1) with a zero-column is equal to zero; it is therefore no loss of generality to suppose that the letters $a, b, c, \ldots$, denote elements of the group S' formed by the non-zero elements of S.

If $roc(n, p) = m$, then the necessary and sufficient condition for that a skewfield S (of characteristic p) is of degree not exceeding n, is

$$[a_0, a_1, \ldots, a_m]\ a_m^{-1} \ldots a_1^{-1}\, a_0^{-1} = 0 \qquad (7)$$

* *l. c.* §§ 3-6.

for every $(m+1)$-tuplet of elements of S'. The left side of (7) is a sum of elements of the commutator group $C(S')$ of S'. Thus the degree of S is characterized by a condition for the elements of $C(S')$.

3. The interconnection between the rank of commutativity of S and the commutator group of S' is not surprising since the condition of commutativity of two elements a and b of S' can be expressed by

$$[a, b] = 0 \quad \text{or by} \quad (a, b) = 1,$$

where

$$(a, b) = aba^{-1}b^{-1}. \tag{8}$$

Furthermore we obtain by elementary calculation:

$$\begin{aligned}[a, b, a^{-1}, b^{-1}] &= (a, b) + (b, a) + (a^{-1}, b^{-1}) + (b^{-1}, a^{-1}) \\ &= (a, b^{-1}) - (b, a^{-1}) - (a^{-1}, b) - (b^{-1}, a). \quad (9)\end{aligned}$$

Introduce for abbreviation

$$[[a, b]] = [[a, b], [a^{-1}, b^{-1}]], \tag{10}$$

then

$$\begin{aligned}[[a, b]] = [[b, a]] &= -[[a^{-1}, b^{-1}]] \\ &= -[[b^{-1}, a^{-1}]] \quad (10') \\ &= (a, b) + (b, a) - (a^{-1}, b^{-1}) - (b^{-1}, a^{-1}). \quad (10'')\end{aligned}$$

We suppose now that $a + b \neq 0$ and put $a + b = c$. Then

$$c^{-1}ab^{-1} = b^{-1} - c^{-1} = b^{-1}ac^{-1}$$

and by the help of this formula we obtain easily:

$$\begin{aligned}&(ba^{-1}, c^{-1}) = (b, a^{-1})\ ;\ (c^{-1}, ba^{-1}) = (a^{-1}, b)\ ; \\ &(ab^{-1}, c^{-1}) = (b, a^{-1})\ ;\ (c^{-1}, ab^{-1}) = (a, b^{-1})\ ; \qquad (11) \\ &(c, ba^{-1}) = c(b, a^{-1})c^{-1}\ ;\ (ba^{-1}, c) = c(a^{-1}, b)c^{-1}\ ; \\ &(ab^{-1}, c) = c(a, b^{-1})c^{-1}\ ;\ (c, ab^{-1}) = c(b^{-1}, a)c^{-1}.\end{aligned}$$

From (9), (10′) and (11) we get

$$\begin{aligned}[c, ab^{-1}, c^{-1}, ba^{-1}] &= c[[a, b^{-1}]]c^{-1} - [[a, b^{-1}]] \\ &= [[c, ab^{-1}]] = [[c, ba^{-1}]]. \quad (12)\end{aligned}$$

Thus the three statements

$$[a+b, ab^{-1}, (a+b)^{-1}, ba^{-1}] = 0$$
$$[[a+b, ab^{-1}]] = [[a+b, ba^{-1}]]$$
$$[[a, b^{-1}]] \text{ is commutative with } a+b \qquad (13)$$

are equivalent.

4. If in particular $[a, b, a^{-1}, b^{-1}] = 0$ for every pair of elements a, b of S', then (13) holds for every pair of elements a, b for which $a+b \neq 0$. On the other hand let (13) be true for every admissible a, b. Put

$$a+b = r,\ ab^{-1} = s;\ \text{then } a = (1-(s+1)^{-1})\,r$$
$$b = (s+1)^{-1}\,r.$$

Therefore, every pair of elements r, s of S' $[r, -1, r^{-1}, -1]$ can be expressed by a and b, unless $s = -1$. Now $[r, -1, r^{-1}, -1] = 0$ and therefore $[a, b, a^{-1}, b^{-1}] = 0$ for every pair of elements of S' when the corresponding statement holds for (13).

Let now S be of degree 2, then $[a, b, c, d] = 0$ for every quadruplet of elements and therefore (13) holds. Hence for every fixed element c of S', $[[c-b, b^{-1}]]$ is commutative with c when b runs over S'. If c does not belong to the centre of S, then the elements of S which are commutative with c, form a subfield F of S which is of rank 2 over the centre and therefore

$$[[c-b, b^{-1}]] = \alpha(b) + \beta(b)c, \qquad (14)$$

where $\alpha(b)$ and $\beta(b)$ are elements of the centre of S. When all the commutators of S' lie in the centre, then $(a, b) = (a^{-1}, b^{-1})$ and therefore $[[a, b]] = 0$ for every a, b. Thus the formulae (13) and (14) are trivial in those cases.

SOME IDENTITIES AND RECURRENTS.

BY

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In this paper are given three recurrents and two pairs of identities of higher differences. The recurrents have been obtained from a factorable determinant resolvable into factors of the type

$$\sum_{x=0}^{k} (-)^x \alpha_x \frac{(a^k-1)(a^{k-1}-1)\ldots(a^{k-x+1}-1)}{(a^x-1)(a^{x-1}-1)\ldots(a-1)} a^{\frac{1}{2}x(x-1)},$$

$$\Sigma = \alpha_0 \text{ if } x = k = 0.$$

1. Notations. If k and c be any numbers and $|k-c| \equiv (\text{mod } b)$, then

$$\begin{bmatrix} k \\ c \end{bmatrix}_b = (a^k-1)(a^{k-b}-1)(a^{k-2b}-1)\ldots(a^c-1),\ a^k-1 \text{ or } 1$$

according as $k > = < c$.

If k and c are positive integers, we write $\begin{bmatrix} k \\ c \end{bmatrix}$ for $\begin{bmatrix} k \\ c \end{bmatrix}_1$. We write ${}^kS_x$ for the sum of the products of k factors $1, a, a^2, a^3, \ldots, a^{k-1}$ taken x at a time. ${}^kS_0 = 1$ and ${}^kS_x = 0$ if $x > k$ or < 0. We also write $a_k = 1 + a^3 + a^5 + \ldots + a^k$, where k is a positive odd integer; $a_k = 1$ if $k = 1$ or -1.

2. Old Theorems.*

$${}^kS_x = \frac{\begin{bmatrix} k \\ 1 \end{bmatrix}}{\begin{bmatrix} x \\ 1 \end{bmatrix}\begin{bmatrix} k-x \\ 1 \end{bmatrix}} {}^xS_x, \qquad (1)$$

* S. Chakrabarti, 'On the evaluation of some factorable continuants' *Bul. Cal. Math. Soc.*, 13 (1922-23), 71-81 and 14 (1923-24), 91-106. "On a Few Algebraic Identities", *ibid*, 27 (1935), 37-44.

where ${}^xS_x$ is evidently $a^{\frac{1}{2}x(x-1)}$;

$$\sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 {}^{2k}S_{2x} = (-)^k; \tag{2}$$

and

$$\sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 {}^{2k+1}S_{1+2x} = (-)^k. \tag{3}$$

3. LEMMAS.

$${}^kS_r \; {}^{k-r}S_m = {}^kS_m \; {}^{k-m}S_r \tag{4}$$

$${}^kS_r = {}^{k-1}S_r + a^{k-1} \; {}^{k-1}S_{r-1}. \tag{5}$$

If $f(x)$ be a function of x or a constant, then

$$\sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 f(x) \; {}^{2k-1}S_{2x}$$

$$= (a^{2k-1}-1) \sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-3-2x \\ 1 \end{bmatrix}_2 f(x) \; {}^{2k-2}S_{2x} \tag{6}$$

$$\sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 f(x) \; {}^{2k}S_{1+2x}$$

$$= (a^{2k}-1) \sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-3-2x \\ 1 \end{bmatrix}_2 f(x) \; {}^{2k-1}S_{1+2x}. \tag{7}$$

[If $f(x) = 1$, the left sides of (6) and (7), by (2) and (3), are $(-)^{k-1}(a^{2k-1}-1)$ and $(-)^{k-1}(a^{2k}-1)$ respectively.]

It is also evident that

$$\sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 f(x) \; {}^{2k-1}S_{2x-1}$$

$$= -\sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-3-2x \\ 1 \end{bmatrix}_2 f(1+x) \; {}^{2k-1}S_{1+2x}. \tag{8}$$

4. THEOREM I.

$$\sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 a^{-2x} \; {}^{2k}S_{2x} = (-)^k(-a^{2k-1}+1+1/a) \tag{9}$$

and

$$\sum_{x=0}^{k} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 a^{-2x}\ {}^{2k+1}S_{1+2x} = (-)^k (-a^{2k+1}+a+1). \tag{10}$$

PROOF. If the left sides of (9) and (10) be denoted by P_k and Q_k respectively, we can easily show that

$$P_k = (a^{2k-1}-1)\ P_{k-1} - a^{2k-3}\ Q_{k-1} \quad \text{by (5), (6) and (8)} \tag{11}$$

and

$$Q_k = (a^{2k}-1)\ Q_{k-1} + a^{2k}\ P_k, \quad \text{by (5) and (7)}$$

$$= a^{2k}(a^{2k-1}-1)\ P_{k-1} + (-a^{4k-3}+a^{2k}-1)\ Q_{k-1} \text{ by (11).}$$

These two relations also hold when P_k and Q_k denote the right sides of (9) and (10). Hence the theorems can be proved by induction.

5. THEOREM 2.

$$\sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 3 \end{bmatrix}_2 a_{2k-1-2x}\ {}^{2k}S_{2x} + (-)^{k-1}\ {}^{2k}S_{2k}$$
$$= (-)^{k-1}({}^{2k}S_1 + {}^{2k-1}S_2\, a^2) \tag{12}$$

and

$$\sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 3 \end{bmatrix}_2 a_{2k-1-2x}\ {}^{2k+1}S_{1+2x} + (-)^{k-1}\ {}^{2k+1}S_{1+2k}$$
$$= (-)^{k-1}({}^{2k+1}S_1 + {}^{2k}S_2\, a^2). \tag{13}$$

PROOF. The left side of (12)

$$= \frac{1}{a-1} \sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 \left(1 + \frac{a^{2k-2x-2}-1}{a^2-1}\, a^3\right) {}^{2k}S_{2x}$$
$$+ (-)^{k-1}\ {}^{2k}S_{2k}$$

$$= \frac{a^2-1-a^3}{(a-1)(a^2-1)} \sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 {}^{2k}S_{2x}$$

$$+ \frac{a^{2k+1}}{(a-1)(a^2-1)} \sum_{x=0}^{k-1} (-)^x \begin{bmatrix} 2k-1-2x \\ 1 \end{bmatrix}_2 a^{-2x}\ {}^{2k}S_{2x} + (-)^{k-1}\ {}^{2k}S_{2k}$$

$$=\frac{a^2-1-a^3}{(a-1)(a^2-1)}\left\{\sum_{x=0}^{k}(-)^x\begin{bmatrix}2k-1-2x\\1\end{bmatrix}_2 {}^{2k}S_{2x}+(-)^{k-1}\,{}^{2k}S_{2k}\right\}$$

$$+\frac{a^{2k+1}}{(a-1)(a^2-1)}\left\{\sum_{x=0}^{k}(-)^x\begin{bmatrix}2k-1-2x\\1\end{bmatrix}_2 a^{-2x}\,{}^{2k}S_{2x}\right.$$

$$\left.+(-)^{k-1}a^{-2k}\,{}^{2k}S_{2k}\right\}+(-)^{k-1}\,{}^{2k}S_{2k}$$

$$=(-)^k\frac{a^2-1-a^3-a^{4k}+a^{2k+1}+a^{2k}}{(a-1)(a^2-1)},\ \text{by (2) and (9)},$$

the cofficient of ${}^{2k}S_{2k}$ being zero,

$$=(-)^{k-1}\left({}^{2k}S_1+{}^{2k-1}S_2\,a^2\right).$$

So (12) is proved and (13) may be similarly proved with the help of (3) and (10).

6. THEOREM 3.

If

$$A_4=\begin{vmatrix}{}^{r+3}S_1 & {}^{r+3}S_2 & {}^{r+3}S_3 & {}^{r+3}S_r\\ 1 & {}^{r+2}S_1 & {}^{r+2}S_2 & {}^{r+2}S_r\\ & 1 & {}^{r+1}S_1 & {}^{r+1}S_r\\ & & 1 & {}^{r}S_r\end{vmatrix}$$

then A_k, *similarly defined*, $=(-)^{\frac{k-1}{2}}\,{}^{r+k-1}S_r\begin{bmatrix}k-2\\1\end{bmatrix}_2$ *or* 0 *according as k is odd or even.*

PROOF. Let us consider the particular case when $k=7$, viz. A_7 which obviously develops into

$$\sum_{x=0}^{5}(-)^x A_{6-x}\,{}^{r+6}S_{1+x}+{}^{r+6}S_r$$

which, if the theorem is true for $A_6, A_5, \ldots A_1$, reduces to

$${}^{r+6}S_r\left\{\sum_{x=0}^{3}(-)^x\begin{bmatrix}5-2x\\1\end{bmatrix}_2 {}^{6}S_{2x}-\begin{bmatrix}5\\1\end{bmatrix}_2+1\right\},\ \text{by (4)},$$

$$=-\,{}^{r+6}S_r\begin{bmatrix}5\\1\end{bmatrix}_2,\ \text{by (2)}.$$

So if the theorem is true for A_6, A_5 etc., it is also true for A_7. The general case may be similarly proved by induction.

7. THEOREM 4.

If

$$B_4 = \begin{vmatrix} {}^6S_1 & {}^6S_2 & {}^6S_3 & {}^6S_4 + {}^6S_2\,{}^3S_1 \\ 1 & {}^5S_1 & {}^5S_2 & {}^5S_3 + {}^5S_2\,{}^2S_1 \\ & 1 & {}^4S_1 & {}^4S_2 + {}^4S_2\,{}^1S_1 \\ & & 1 & {}^3S_1 \end{vmatrix}$$

then, B_k*, similarly formed* $= (-)^p \frac{{}^{k+2}S_2}{{}^2S_2}\begin{bmatrix} k \\ 3 \end{bmatrix}_2$ *or* $(-)^p \frac{{}^{k+2}S_2}{{}^2S_2}\begin{bmatrix} k-1 \\ 1 \end{bmatrix}_2$*, according as* k *is odd or even,* p *being the integral part of* $k/2$.

PROOF. Let us consider the recurrent B_6 which develops into

$$\sum_{x=0}^{5} (-)^x\, {}^8S_{1+x}\, B_{5-x} - {}^8S_2\, {}^5S_1, \quad (B_0 = 1)$$

which, if the theorem is true for $B_5, B_4, \ldots B_1$, reduces to

$$\frac{{}^8S_2}{{}^2S_2}\left\{\frac{1}{a-1}\sum_{x=0}^{2}(-)^x\begin{bmatrix} 5-2x \\ 1 \end{bmatrix}_2 {}^6S_{1+2x} + \sum_{x=0}^{3}(-)^x\begin{bmatrix} 5-2x \\ 1 \end{bmatrix}_2 {}^6S_{2x} - \begin{bmatrix} 5 \\ 1 \end{bmatrix}_2 - {}^5S_1\, a\right\} \text{ by (4).}$$

$$= \frac{{}^8S_2}{{}^2S_2}\left\{\frac{a^6-1}{a-1} + (-)^3 - \begin{bmatrix} 5 \\ 1 \end{bmatrix}_2 - {}^5S_1 a\right\}, \text{ by (7) and (2)}$$

$$= -\frac{{}^8S_2}{{}^2S_2}\begin{bmatrix} 5 \\ 1 \end{bmatrix}_2.$$

So the theorem holds in the case of B_6, if it is true for $B_5, B_4 \ldots B_1$. The general case may be similarly proved by induction.

8. THEOREM 5.

If

$$C_4 = \begin{vmatrix} {}^7S_1 & {}^7S_2 & {}^7S_3 & {}^7S_1\,{}^5S_2\, a^2 + (-)^4\, {}^7S_1 \\ 1 & {}^6S_1 & {}^6S_2 & {}^6S_1\,{}^4S_2\, a^2 \\ & 1 & {}^5S_1 & {}^5S_1\,{}^3S_2\, a^2 \\ & & 1 & {}^4S_1\,{}^2S_2\, a^2 \end{vmatrix}$$

then, $C_k = (-)^p\, {}^{k+3}S_1 \begin{bmatrix} k+2 \\ 3 \end{bmatrix}_2$ *or* $(-)^p\, {}^{k+3}S_1 \begin{bmatrix} k+1 \\ 3 \end{bmatrix}_2 a_{k+1}$, *according as k is odd or even, p being the integral part of* $\frac{k-1}{2}$.

PROOF. Let us consider the recurrent C_8, which develops into

$$\left\{\sum_{x=0}^{6} (-)^x \left(C_{7-x} + {}^{10-x}C_1\right) {}^{11}S_{1+x}\right\} - {}^{11}S_1\, {}^9S_2\, a^2 - {}^{11}S_1$$

which, if the theorem is true for $C_7, C_6, \ldots C_1$, reduces, by (4), to

$$ {}^{11}S_1 \Bigg[-\frac{1}{a-1} \sum_{x=0}^{5-1} (-)^x \begin{bmatrix} 9-2x \\ 1 \end{bmatrix}_2 {}^{10}S_{1+2x}$$

$$+ \left\{ \sum_{x=0}^{5-1} (-)^x \begin{bmatrix} 9-2x \\ 3 \end{bmatrix}_2 a_{9-2x}\, {}^{10}S_{2x} + (-)^4\, {}^{10}S_{10} \right\} - \begin{bmatrix} 9 \\ 3 \end{bmatrix}_2 a_9 - {}^9S_2\, a^2 \Bigg]$$

$$\text{for } \sum_{x=0}^{k} (-)^x\, {}^kS_x = 0$$

$$= {}^{11}S_1 \left\{ -\frac{a^{10}-1}{a-1} (-)^4 + (-)^4 \left({}^{10}S_1 + {}^9S_2\, a^2\right) \right.$$

$$\left. - \begin{bmatrix} 9 \\ 3 \end{bmatrix}_2 a_9 - {}^9S_2\, a^2 \right\}, \text{ by (7) and (12)}$$

$$= -\,{}^{11}S_1 \begin{bmatrix} 9 \\ 3 \end{bmatrix}_2 a_9.$$

Hence if the theorem holds in each of the cases of $C_7, C_6, \ldots C_1$, it also holds in the case of C_8. The general case may be similarly proved by induction.

ON FRULLANI INTEGRALS.

BY

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A necessary and sufficient condition for a Frullani integral to exist has been given by K. S. K. Iyengar* and it may be stated as follows:

A necessary and sufficient condition for the existence of

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\varepsilon p} \frac{\phi(t)}{t} \, dt$$

as a finite limit is that the function

$$\Phi(x) = \int_0^x \phi(t) \, dt$$

be determinate for $0 \leqslant x < +\infty$ *and that* $\Phi'(+0)$ *exist and be finite.*

The proof given by K. S. K. Iyengar is rather complicated. A much simpler proof will be developed in this paper.

First of all, K. S. K. Iyengar shows that the existence of

$$-\infty < \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\varepsilon p} \frac{\phi(t)}{t} \, dt < +\infty$$

implies the existence of

$$\int_0^x \phi(t) \, dt, \; 0 \leqslant x < +\infty;$$

whenever

$$\int_{\alpha}^x \phi(t) \, dt, \; 0 < \alpha < +\infty$$

has a meaning.

The proof which he gives seems the most direct and simple that can be found.

Now, by a classical formula of calculus,

* *Journ. Ind. Math. Soc.*, (2) 4 (1940), 145-50.

$$\int_{\varepsilon}^{\varepsilon\rho} \frac{\phi(t)}{t}\, dt = [\psi(t)]_{\varepsilon}^{\varepsilon\rho} + \int_{\varepsilon}^{\varepsilon\rho} \frac{\psi(t)}{t}\, dt,\ \psi(t) = \frac{1}{t}\int_0^t \phi(u)\, du. \quad (1)$$

If $\lim_{\varepsilon\to 0} \psi(t) = \Phi'(+0)$ exists, then

$$\lim_{\varepsilon\to 0} [\psi(t)]_{\varepsilon}^{\varepsilon\rho} = 0$$

and by (1),

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\varepsilon\rho} \frac{\phi(t)}{t}\, dt = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\varepsilon\rho} \frac{\psi(t)}{t}\, dt = \Phi'(+0) \log \rho.$$

Further we have

$$\int_0^u dx \int_1^\rho \frac{\phi(xy)}{y}\, dy = \int_1^\rho dy \int_0^u \frac{\phi(xy)}{y}\, dx = u\int_u^{\rho u} \frac{\psi(t)}{t}\, dt, \quad (2)$$

for this equation may be written as

$$\int_0^u dx \left\{ \left[\frac{\Phi(t)}{t} \right]_x^{x\rho} + \int_x^{x\rho} \Phi(t)\, \frac{dt}{t^2} \right\} = u\int_u^{u\rho} \Phi(t)\, \frac{dt}{t^2},\ \Phi(t) = \int_0^t \phi(t)\, dt,$$

where $\Phi(t)$ is continuous for $0 \leqslant t < +\infty$ and, in this last form is easily verified.

It follows that if

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\varepsilon\rho} \frac{\phi(t)}{t}\, dt = l,\ |l| < +\infty,$$

then

$$\lim_{u\to 0} \frac{1}{u}\int_0^u dx \int_x^{x\rho} \phi(t)\, \frac{dt}{t} = l.$$

By (2) we have

$$l = \lim_{u\to 0}\int_u^{u\rho} \frac{\psi(t)}{t}\, dt.$$

Hence, from (1), we get

$$\lim_{\varepsilon\to 0} [\psi(t)]_{\varepsilon}^{\varepsilon\rho} = 0. \quad (3)$$

Now it must be shown that (3) implies

$$\lim_{\varepsilon\to 0} \psi(\varepsilon) = \Phi'(+0). \quad (4)$$

In fact, for $t > 0$, $\psi(t)$ is continuous. Hence, if $0 < r < 1$,

$$\int_{r^n u}^{r^{n-1}u} \psi(t)\, \frac{dt}{t} = \psi(u_n) \log \frac{1}{r},\quad r^n u < u_n < r^{n-1} u$$

and the sequence $\{\psi(u_n)\}$, $n = 1, 2, \ldots$ is convergent, converging to a finite limit, L say. By (3) the sequence $\{\psi(\rho u_n)\}$, $\rho > 0$, $\rho \neq 1$, is also convergent converging to L.

Let $\{v_n\}$ be a monotonic, decreasing, sequence of positive numbers converging to zero. There is no loss of generality in assuming that $r^n u \leqslant v_n \leqslant r^{n-1} u$ so that $r \leqslant v_n/u_n \leqslant 1/r$. Let $v_m = \rho_m u_m$, $m = 1, 2, \ldots$.

For every positive integer m we have $\lim_{n\to\infty} \psi(\rho_m u_n) = L$.

Therefore

$$\lim_{n\to\infty} \psi(\rho_n u_n) = L,$$

and as the sequence $\{v_n\}$ is arbitrary, (4) follows.

The converse of this theorem is easily proved. It may be stated as follows:

A necessary and sufficient condition for $\Phi'(+0)$ *to exist as a finite limit, where*

$$\Phi(x) = \int_0^x \psi(t)\, dt$$

is that

$$\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\varepsilon\rho} \psi(t) \frac{dt}{t} = l, \quad |l| < +\infty$$

for any positive number ρ, $\rho \neq 1$.